# Lecture 16: Integration - I

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# Revisiting Green's Theorem

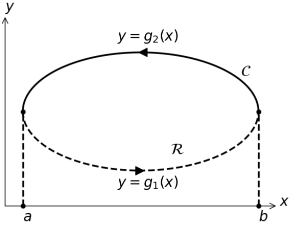
Let $u$ and $v$ be analytic in some $\mathcal{R}$ bounded by a simple curve $\mathcal{C}$

$$\oint_{\mathcal{C}} (u \, \mathrm{d}x + v \, \mathrm{d}y) = \iint_{\mathcal{R}} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y$$

$\mathcal{C}$ is traversed counter-clockwise (positive orientation)

**Proof**

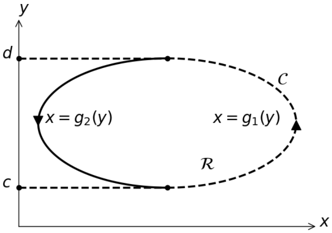
In the region we first work out




$$
\begin{aligned}
& - \iint_{\mathcal{R}} \frac{\partial u}{\partial y} \, \mathrm{d}x \, \mathrm{d}y \\
= & - \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \frac{\partial u}{\partial y} \, \mathrm{d}y \, \mathrm{d}x \\
= & - \int_{x=a}^{x=b} [u(x, g_2(x)) - u(x, g_1(x))] \, \mathrm{d}x \\
= & \oint_{\mathcal{C}} u \, \mathrm{d}x
\end{aligned}
$$

# Green's Theorem

In the same region we now show




$$\iint_{\mathcal{R}} \frac{\partial v}{\partial x}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{y=c}^{y=d} \int_{x=g_2(y)}^{x=g_1(y)} \frac{\partial v}{\partial x}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{y=c}^{y=d} [v(g_1(y), y) - v(g_2(y), y)]\,\mathrm{d}y$$

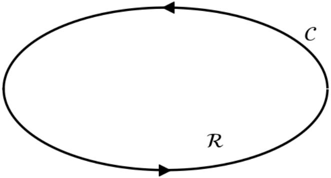
$$= \oint_{\mathcal{C}} v\,\mathrm{d}y$$

Giving us the Green's theorem

$$\boxed{\oint_{\mathcal{C}} (u\,\mathrm{d}x + v\,\mathrm{d}y) = \iint_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y}$$

# Cauchy's Theorem

If $f(z)$ is analytic in some region $\mathcal{R}$ bounded by a simple curve $\mathcal{C}$




$$\boxed{\oint_{\mathcal{C}} f(z)\, \mathrm{d}z = 0}$$

**Proof**

$$\begin{aligned}
\oint_{\mathcal{C}} f(z)\, \mathrm{d}z &= \oint_{\mathcal{C}} (u + iv)\, (\mathrm{d}x + i\, \mathrm{d}y) \\
&= \oint_{\mathcal{C}} (u\, \mathrm{d}x - v\, \mathrm{d}y) + i \oint_{\mathcal{C}} (u\, \mathrm{d}y + v\, \mathrm{d}x) \\
&= \iint_{\mathcal{R}} \left[ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + i \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \right] \mathrm{d}x\, \mathrm{d}y \qquad \ldots \mathrm{GT} \\
&= 0 \qquad \ldots \mathrm{CR}
\end{aligned}$$

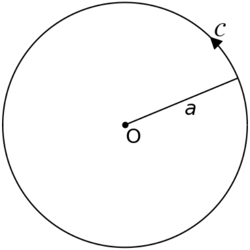
# Converse is True!

If $\oint_{\mathcal{C}} f(z)\, \mathrm{d}z = 0$ and $f$ is continuous everywhere inside $\mathcal{C}$, then $f$ is also analytic inside $\mathcal{C}$.

Proof left as exercise

**Hint:** Just use the fact that $u, v$ must be continuous along with their partial derivatives for Green's theorem to apply.
Cauchy-Riemann conditions will naturally follow from there.

# Simple Example




Evaluate $\mathcal{I} = \oint_{\mathcal{C}} z^n \, \mathrm{d}z$ with $n \in \mathbb{Z}$

With $z = a\, e^{i\theta}$

$$\mathcal{I} = \oint_{\mathcal{C}} z^n \, \mathrm{d}z = \int_0^{2\pi} a^{n+1} e^{i(n+1)\theta} i \, \mathrm{d}\theta = \begin{cases} 2\pi i & (n = -1) \\ 0 & (n \neq -1) \end{cases}$$

The answer is also independent of $a$!

**Food for thought**

**Q.** $\mathcal{I}$ vanishes for $\frac{1}{z^2}, \frac{1}{z^3} \ldots$ These functions are not analytic in $\mathcal{R}$

**A.** The converse of Cauchy theorem is not applicable as none of these functions are continuous at origin

# Generalization

We could have taken the circle origin anywhere, say $z_0$. Then, the parametrization $z - z_0 = ae^{i\theta}$ gives

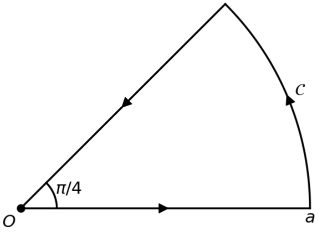
$$\oint_{\mathcal{C}} (z - z_0)^n \, \mathrm{d}z = \oint_{\theta}^{2\pi} a^n e^{in\theta} a i e^{i\theta} \, \mathrm{d}\theta = \begin{cases} 2\pi i & (n = -1) \\ 0 & (n \neq -1) \end{cases}$$

As usual, the integral does not depend on the circle radius $a$.

# Application in Electromagnetism

Evaluate the Fresnel integral (very hard by elementary method)

$$\mathcal{I} = \int_0^\infty e^{ix^2} \mathrm{d}x$$




By Cauchy theorem,

$$\oint_{\mathcal{C}} e^{iz^2}\, \mathrm{d}z = 0$$

gives the sum of three line integrals,

$$\int_0^a e^{ix^2} \mathrm{d}x + i\, a \int_0^{\pi/4} e^{ia^2 \cos 2\theta} e^{-a^2 \sin 2\theta}\, e^{i\theta}\, \mathrm{d}\theta + e^{i\pi/4} \int_a^0 e^{-r^2} \mathrm{d}r = 0$$

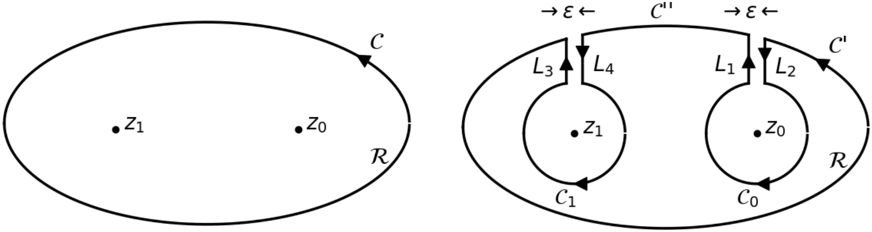
In the limit $a \to \infty$

$$\mathcal{I} + e^{i\pi/4} \int_\infty^0 e^{-r^2} \mathrm{d}r = 0$$

$$\boxed{\mathcal{I} = e^{i\pi/4} \sqrt{\pi}/2}$$

# Contour Deformation

How to evaluate $\oint_{\mathcal{C}} f(z)\,\mathrm{d}z$ with two singularities $z_0$ and $z_1$ inside?




Given

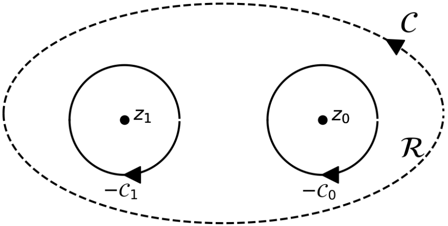
Cut and deform

Since $z_{0,1}$ are outside the deformed contour, Cauchy's formula gives

$$\int_{\mathcal{C}'} + \int_{\mathcal{L}_2} + \int_{\mathcal{L}_1} + \int_{\mathcal{C}_0} + \int_{\mathcal{C}''} + \int_{\mathcal{L}_4} + \int_{\mathcal{L}_3} + \int_{\mathcal{C}_1} f(z)\,\mathrm{d}z = 0$$

# Stitching

In the limit $\epsilon \to 0$, we get




Two new circles with $-$ve orientation

and the integral reduces to

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z + \underbrace{\oint_{-\mathcal{C}_1} f(z)\,\mathrm{d}z + \oint_{-\mathcal{C}_0} f(z)\,\mathrm{d}z}_{-\text{ve orientation}} = 0$$

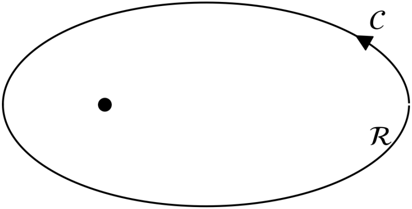
Put simply

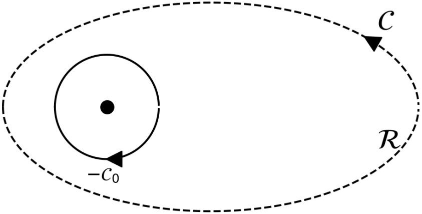
$$\boxed{\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = \oint_{\mathcal{C}_1} f(z)\,\mathrm{d}z + \oint_{\mathcal{C}_0} f(z)\,\mathrm{d}z}$$

# Worked Example

Evaluate $\displaystyle\oint_{\mathcal{C}} \frac{e^{z^2}}{z^2}\,\mathrm{d}z$ on some contour $\mathcal{C}$ about the origin.

Solution: Clearly we cannot use the Cauchy's theorem as $e^{z^2}/z^2$ is not analytic at the origin. Below we apply the trick just learned




Deform some contour $\mathcal{C}$ to a circle about origin

$$\oint_{\mathcal{C}} \frac{e^{z^2}}{z^2}\,\mathrm{d}z = \oint_{\mathcal{C}_0} \frac{e^{z^2}}{z^2}\,\mathrm{d}z = \oint_{\mathcal{C}_0} \left( \frac{1}{z^2} + 1 + \frac{z^2}{2!} + \dots \right) \mathrm{d}z = 0$$