Lecture 17: Integration - II

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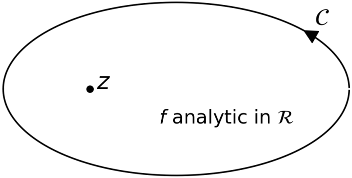
Cauchy's Integral Formula

An f is analytic in some region $\mathcal{R}$ bounded by some $\mathcal{C}$, then anywhere inside $\mathcal{R}$, it is determined by the boundary integral



$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{z - \xi} \, \mathrm{d}\xi$$

Proof

By deforming the contour $\mathcal{C}$ to a circle of radius δ and center at z

$$\begin{aligned} \oint_{\mathcal{C}} \frac{f(\xi)}{z-\xi} \, \mathrm{d}\xi &= \oint_{\mathcal{C}_\delta} \frac{f(z)}{z-\xi} \, \mathrm{d}\xi - \oint_{\mathcal{C}_\delta} \frac{f(z) - f(\xi)}{z-\xi} \, \mathrm{d}\xi \\ &= 2\pi i \, f(z) - \oint_{\mathcal{C}_\delta} \frac{f(z) - f(\xi)}{z-\xi} \, \mathrm{d}\xi \end{aligned}$$

In the limit $\delta \to 0$, the second integral becomes $f'(z) \oint_{\mathcal{C}_\delta} \mathrm{d}\xi \to 0$

Cauchy's Differential Formula

Analytic f is infinitely differentiable inside $\mathcal{R}$ with the n^{th} derivative

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^{n+1}} \, \mathrm{d}\xi} \qquad z \in \mathcal{R} \quad \xi \in \mathcal{C}$$

Proof

Since $f(z)$ is analytic, $f'(z)$ exists (or unique)

$$f''(z) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi\right] = \frac{2!}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^3} \, \mathrm{d}\xi \quad \text{exists}$$

Thus $f'(z)$ is analytic

$$f'''(z) = \frac{\mathrm{d}^3}{\mathrm{d}z^3}\left[\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi\right] = \frac{3!}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^4} \, \mathrm{d}\xi \quad \text{exists}$$

By induction all derivatives are established

Evaluate $\mathcal{I} = \oint_{\mathcal{C}} \frac{\sin z}{(z-\pi/2)^3}\, \mathrm{d}z$, where $\mathcal{C}$ encloses $z = \pi/2$

Solution

Recall that $\sin z$ is an entire. From Cauchy's differential formula

$$\mathcal{I} = \frac{2\pi i}{2!} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \sin z \bigg|_{\pi/2} = -\pi i$$

Alternatively, from cosine series

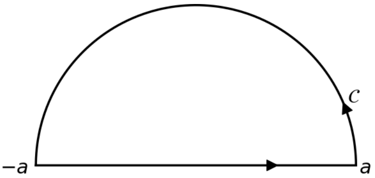
$$\mathcal{I} = \oint_{\mathcal{C}} \frac{\cos(z-\pi/2)}{(z-\pi/2)^3}\, \mathrm{d}z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \oint_{\mathcal{C}} (z-\pi/2)^{2n-3}\, \mathrm{d}z = -\pi i$$

Worked Example

Evaluate $\mathcal{I} = \int_{-\infty}^{\infty} \frac{1}{(x+i)^2}\, \mathrm{d}x$

Solution

Take the semicircle in plane $y > 0$



$$\oint_C \frac{1}{(z+i)^2}\, \mathrm{d}z = \int_{-a}^{a} \frac{1}{(x+i)^2}\, \mathrm{d}x + \int_0^{\pi} \frac{aie^{i\theta}}{a^2e^{2i\theta} + 2ae^{i\theta}i - 1}\, \mathrm{d}\theta = 0$$

Now $\left| \frac{aie^{i\theta}}{a^2e^{2i\theta} + 2ae^{i\theta}i - 1} \right| \leq \underbrace{\frac{a}{a^2 - 2a - 1}}_{\text{max value}} \longrightarrow 0 \quad \text{as} \quad a \longrightarrow \infty$

Giving us $\int_{-\infty}^{\infty} \frac{1}{(x+i)^2}\, \mathrm{d}x = 0$

Worked Example

Evaluate $\mathcal{I} = \oint_{\mathcal{C}} \frac{z}{z^2 - 1/9}\,\mathrm{d}z$, where $\mathcal{C}$ is a unit circle at O

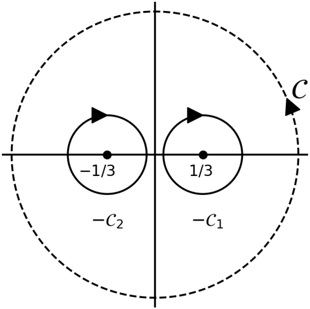
Solution

Deforming into two small circles $\mathcal{C}_{1,2}$

$$\mathcal{I} = \oint_{\mathcal{C}_1} + \oint_{\mathcal{C}_2} \left[\frac{z}{(z - 1/3)(z + 1/3)} \right] \mathrm{d}z$$

$$= 2\pi i \left[\frac{1}{2} + \frac{1}{2} \right]$$

$$= 2\pi i$$



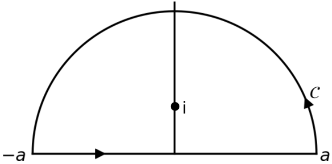
Worked Example

Evaluate $\mathcal{I} = \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

Solution

On the contour $\mathcal{C}$, compute

$$\oint_{\mathcal{C}} \frac{1}{1+z^2}\,\mathrm{d}z = \oint_{\mathcal{C}} \frac{\mathrm{d}z}{(z+i)(z-i)} = \pi$$



But the contour integral

$$\oint_{\mathcal{C}} \frac{1}{1+z^2}\,\mathrm{d}z = \int_{-a}^{a} \frac{1}{1+x^2}\,\mathrm{d}x + \int_0^\pi \frac{aie^{i\theta}}{1+a^2e^{i2\theta}}\,\mathrm{d}\theta$$

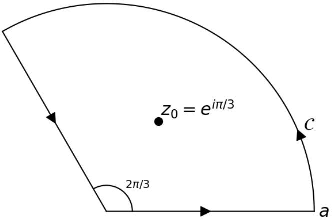
The second integral vanishes in the limit $a \to \infty$, yielding

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = 2\mathcal{I} = \pi$$

Worked Example

Evaluate $\mathcal{I} = \int_0^\infty \frac{1}{1+x^3}\,\mathrm{d}x$

Solution



$$\text{Take } \mathcal{J} = \oint_{\mathcal{C}} \frac{1}{1+z^3}\,\mathrm{d}z = \frac{2\pi i}{(e^{i\pi/3}-e^{i\pi})(e^{i\pi/3}-e^{i5\pi/3})} = \frac{2\pi i}{3e^{2\pi i/3}}$$

But from the figure, we see that

$$\mathcal{J} = \int_0^a \frac{1}{1+x^3}\,\mathrm{d}x + \int_0^{2\pi/3} \frac{iae^{i\theta}}{1+a^3e^{i3\theta}}\,\mathrm{d}\theta + \int_a^0 \frac{e^{i2\pi/3}}{1+r^3}\,\mathrm{d}r$$

Equating the above two in the limit $a \to \infty$, we get

$$\mathcal{I} = \frac{2\pi i}{3e^{2\pi i/3}}\,\frac{1}{1-e^{i2\pi/3}} = \frac{2\pi}{3\sqrt{3}}$$