Lecture 23: Generalizing Fourier Series

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Extension to Arbitrary Periods

- Fourier series can be used to represent any L-periodic function
- This is done by letting

$$x = \frac{\theta L}{2\pi}$$

- The interval of 2π in θ becomes an interval of L in x
- The formulas for Fourier series become

$$\boxed{f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2\pi nx}{L} + B_n \sin\frac{2\pi nx}{L}\right)}$$

with

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\frac{2\pi mx}{L}\,\mathrm{d}x$$

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi mx}{L}\,\mathrm{d}x$$

Complex Notation

By using the Euler representation

$$e^{i2\pi nx/L} = \cos\frac{2\pi nx}{L} + i\sin\frac{2\pi nx}{L}$$

We can rewrite our Fourier series in the complex form

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} a_n\, e^{i2\pi nx/L}}$$

To get a_m, we multiply by $e^{-i2\pi mx/L}$ and integrate from 0 to L

$$\frac{1}{L}\int_0^L f(x)\, e^{-i2\pi mx/L}\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} a_n \underbrace{\left(\frac{1}{L}\int_0^L e^{i2\pi(n-m)x/L}\,\mathrm{d}x\right)}_{\delta_{nm}} = a_m$$

Average Absolute Square

Very useful in physics applications, and is obtained as

$$\frac{1}{L}\int_0^L |f(x)|^2 \, \mathrm{d}x = \frac{1}{L}\int_0^L \left(\sum_{n=-\infty}^{\infty} a_n \, e^{i2\pi nx/L}\right)\left(\sum_{m=-\infty}^{\infty} a_m^* \, e^{-i2\pi mx/L}\right) \mathrm{d}x$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n a_m^* \underbrace{\left(\frac{1}{L}\int_0^L e^{i2\pi(n-m)x/L} \, \mathrm{d}x\right)}_{\delta_{nm}}$$

yielding,

$$\boxed{\frac{1}{L}\int_0^L |f(x)|^2 \, \mathrm{d}x = \sum_{n=-\infty}^{\infty} |a_n|^2}$$

“Each Fourier mode contributes independently to the integral”

Fourier Transforms

We begin with our Fourier series for L-periodic function

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \, e^{i2\pi nx/L} \qquad a_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-i2\pi nx/L}\, \mathrm{d}x$$

As $L \to \infty$, f becomes aperiodic and is defined on the full x-axis

The sum is converted to an integral by defining,

$$\frac{2\pi n}{L} = k \quad \text{and} \quad a_n = \frac{\tilde{f}(k)}{L}$$

Since n increases in steps of unity

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \, e^{i2\pi nx/L} = \int_{-\infty}^{\infty} a_n \, e^{i2\pi nx/L}\, \mathrm{d}n = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx}\, \mathrm{d}k$$

Transform Pairs

Put simply

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, \mathrm{d}k \qquad \tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x}$$

is the Fourier transform pair. The position of $\frac{1}{2\pi}$ is also arbitrary

One can use a symmetric form also

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, \mathrm{d}k \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x}$$

Replacing $i \to -i$ in the above does not affect the transform

The Dirac δ-distribution

Fourier transforms provide an integral representation of $\delta(x)$

From the definition of transforms

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, \mathrm{d}k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(x')\, e^{-ikx'}\, \mathrm{d}x'}_{\tilde{f}(k)}\; e^{ikx}\, \mathrm{d}k \\
&= \int_{-\infty}^{\infty} f(x')\; \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\, \mathrm{d}k}_{?}\; \mathrm{d}x' \\
&= \int_{-\infty}^{\infty} f(x')\, \delta(x-x')\, \mathrm{d}x' \qquad \text{...definition of } \delta(x)
\end{aligned}
$$

Thus,

$$
\boxed{\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\, \mathrm{d}k}
$$

Parseval's theorem

Absolute square integral of f remains invariant in x- and k-space

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k$$

Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x &= \int_{-\infty}^{\infty} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx} \, \mathrm{d}k}_{f(x)} \; \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}^*(k')\, e^{-ik'x} \, \mathrm{d}k'}_{f^*(x)} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \int_{-\infty}^{\infty} \tilde{f}^*(k') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} \, \mathrm{d}x \right) \mathrm{d}k' \, \mathrm{d}k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \underbrace{\int_{-\infty}^{\infty} \tilde{f}^*(k')\, \delta(k-k') \, \mathrm{d}k'}_{?} \, \mathrm{d}k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k)\, \tilde{f}^*(k) \, \mathrm{d}k = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k
\end{aligned}$$