Lecture 24: Fourier Transforms

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Useful Formulas

- An L-periodic function $f(x)$ becomes aperiodic as $L \to \infty$
- $f(x)$ now defined on the full x-axis, is represented by the pair

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, \mathrm{d}k$$

- Average absolute square is an invariant- Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, \mathrm{d}k$$

- Fourier transform of unity is the Dirac δ-distribution

$$\int_{-\infty}^{\infty} e^{-ikx}\, \mathrm{d}x = 2\pi\, \delta(k) \qquad \int_{-\infty}^{\infty} e^{ikx}\, \delta(k)\, \mathrm{d}k = 1$$

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Symmetries of $f(x)$

An <u>even</u> $f(x)$ is represented by cosine transforms

$$\begin{aligned}
\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x &= \int_0^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x + \underbrace{\int_{-\infty}^{0} f(x)\, e^{-ikx}\, \mathrm{d}x}_{x \to -x} \\
&= \int_0^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x + \int_0^{\infty} f(x)\, e^{ikx}\, \mathrm{d}x \\
&= 2\int_0^{\infty} f(x) \cos kx\, \mathrm{d}x
\end{aligned}$$

Since $\tilde{f}(k)$ is also even, we write the inverse using above method

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \tilde{f}(k) \cos kx\, \mathrm{d}k$$

Symmetries of $f(x)$

An odd $f(x)$ is represented by sine transforms

$$\begin{aligned}\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x &= \int_0^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x + \underbrace{\int_{-\infty}^{0} f(x)\, e^{-ikx}\, \mathrm{d}x}_{x \to -x} \\ &= \int_0^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x - \int_0^{\infty} f(x)\, e^{ikx}\, \mathrm{d}x \\ &= -2i \int_0^{\infty} f(x) \sin\, kx\, \mathrm{d}x\end{aligned}$$

Since $\tilde{f}(k)$ is now odd, we write the inverse using above method

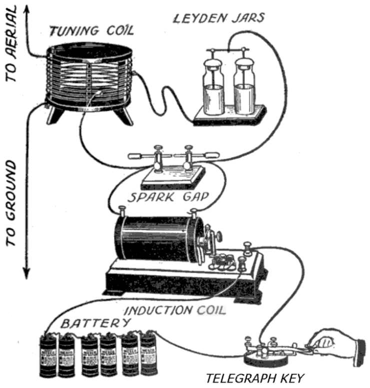
$$f(x) = \frac{i}{\pi} \int_0^{\infty} \tilde{f}(k) \sin\, kx\, \mathrm{d}k$$

We can get rid of i in the above pair by redefining

$$\tilde{f}(k) = 2 \int_0^{\infty} f(x) \sin\, kx\, \mathrm{d}x \qquad f(x) = \frac{1}{\pi} \int_0^{\infty} \tilde{f}(k) \sin\, kx\, \mathrm{d}k$$

Application in Radio Transmission

A spark gap transmitter (WW-II) emitting radio waves



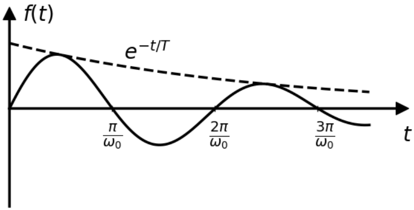
The electric field has a magnitude

$$f(t) = \begin{cases} e^{-t/T} \sin \omega_0 t & (t > 0) \\ 0 & (t < 0) \end{cases}$$

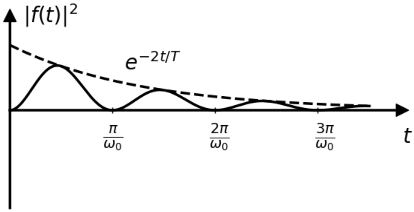
1. Find the total radiated energy
2. Derive and sketch the energy spectrum in frequency domain

Sketches

Damped sine wave



Corresponding power



Total Radiated Energy

Total radiated energy is proportional to

$$\int_0^\infty |f(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 \, \mathrm{d}\omega \quad \ldots \text{Parseval's theorem}$$

We use the time domain to compute energy

$$\begin{aligned}
\int_0^\infty |f(t)|^2 \, \mathrm{d}t &= \int_0^\infty e^{-2t/T} \sin^2 \omega_0 t \, \mathrm{d}t \\
&= -\frac{1}{4} \int_0^\infty e^{-2t/T} \left(e^{2i\omega_0 t} + e^{-2i\omega_0 t} - 2\right) \mathrm{d}t \\
&= -\frac{1}{4} \left[\frac{e^{-2t/T + 2i\omega_0 t}}{(-2/T + 2i\omega_0)} + \frac{e^{-2t/T - 2i\omega_0 t}}{(-2/T - 2i\omega_0)} + T \, e^{-2t/T} \right]_0^\infty \\
&= \frac{1}{4} \left[\frac{1}{(-2/T + 2i\omega_0)} + \frac{1}{(-2/T - 2i\omega_0)} + T \right] \\
&= \frac{T}{4} \left[\frac{\omega_0^2}{1/T^2 + \omega_0^2} \right]
\end{aligned}$$

Frequency Domain

Fourier transform of the radiated wave

$$
\begin{aligned}
\tilde{f}(\omega) &= \int_0^\infty f(t)\, e^{-i\omega t}\, \mathrm{d}t \\
&= \int_0^\infty e^{-t/T} \sin \omega_0 t\, e^{-i\omega t}\, \mathrm{d}t \\
&= \frac{1}{2i} \int_0^\infty e^{-t/T}\, (e^{i\omega_0 t} - e^{-i\omega_0 t})\, e^{-i\omega t}\, \mathrm{d}t \\
&= \frac{1}{2i} \left[\frac{e^{-t/T + i\omega_0 t - i\omega t}}{(-1/T + i\omega_0 - i\omega)} - \frac{e^{-t/T - i\omega_0 t - i\omega t}}{(-1/T - i\omega_0 - i\omega)} \right]_0^\infty \\
&= \frac{1}{2} \left[\frac{1}{\omega + \omega_0 - i/T} - \frac{1}{(\omega - \omega_0 - i/T)} \right]
\end{aligned}
$$

is a complex quantity with $\tilde{f}(\omega) \neq \tilde{f}(-\omega)$

Energy Spectrum

- A physically measurable quantity is the energy spectrum

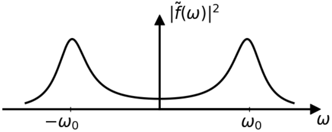
$$|\tilde{f}(\omega)|^2 = \tilde{f}(\omega)\ \tilde{f}^*(\omega) = \frac{\omega_0^2}{(1/T^2 + \omega_0^2 - \omega^2)^2 + 4\omega^2/T^2}$$

which is the energy radiated per unit frequency interval

- Notice $|\tilde{f}(\omega)|^2 = |\tilde{f}(-\omega)|^2$: symmetric about $\omega = 0$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2\, d\omega = \frac{1}{\pi}\int_0^{\infty} |\tilde{f}(\omega)|^2\, d\omega$$

negative frequencies are not required to compute total energy!



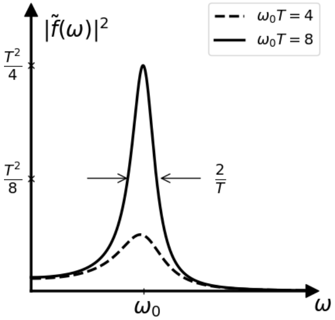
Peak power at $\pm\omega_0$, the natural frequency

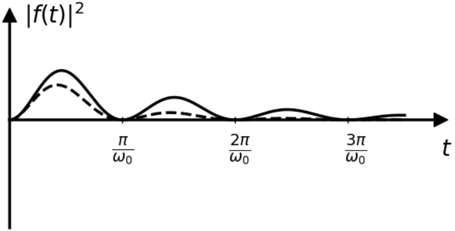
Uncertainty Principle

In the weak damping limit or large T, i.e $\omega_0 T \gg 1$

The spectrum must be sharply peaked near $\omega = \pm\omega_0$. Near ω_0,

$$\tilde{f}(\omega) \approx -\frac{1}{2}\frac{1}{\omega - \omega_0 - i/T} \implies |\tilde{f}(\omega)|^2 \approx \frac{1}{4}\frac{1}{(\omega-\omega_0)^2 + 1/T^2}$$





uncertainty in frequency, FWHM $\propto \frac{1}{T}$