Lecture 26: Laplace Transforms

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Motivation

- There are useful functions that cannot be Fourier transformed
- Simple examples, $f(x) = \text{const}$, and $f(x) = x^n$ $(n > 0)$
- Their Fourier transformation integrals do not converge
- A quickfix is to multiply $f(x)$ by a factor $e^{-\alpha x}$ with $\alpha > 0$
- But this will spoil convergence in the limit $x \to -\infty$
- Last handicap can be removed if we need $f(x)$ only for $x > 0$
- We thus multiply $f(x)$ by the Heavyside function

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\underbrace{f(x)}_{\text{not Fourier transformable}} \xrightarrow{\text{is transformed to}} \underbrace{f(x)\,\Theta(x)\,e^{-\alpha x}}_{\text{Fourier transformable}}$$

Laplace Transforms

We can now construct a transform pair

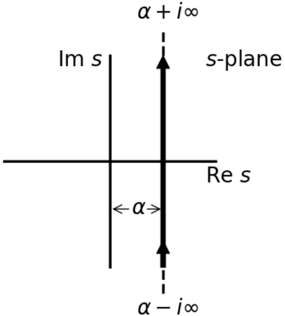
$$\tilde{g}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-\alpha x}\, \Theta(x)\, e^{-ikx}\, \mathrm{d}x$$

$$f(x)\, e^{-\alpha x}\, \Theta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(k)\, e^{ikx}\, \mathrm{d}k$$

Take $s = \alpha + ik$, and define $\tilde{g}(k) \equiv F(s)$. The above pair becomes

$$F(s) = \int_{0}^{\infty} f(x)\, e^{-sx}\, \mathrm{d}x$$

$$\underbrace{f(x) \;= \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} F(s)\, e^{sx}\, \mathrm{d}s}_{\text{Bromwich integral}}$$



$F(s)$ exists only on the plane, Re $s > \alpha$

Worked Example

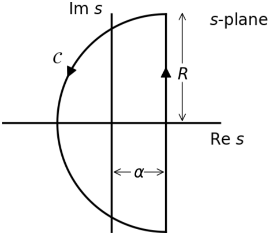
Compute the Laplace transform of $f(x) = 1$, and verify by inversion

Solution

$$F(s) = \int_0^\infty f(x)\, e^{-sx}\, \mathrm{d}x = \int_0^\infty e^{-sx}\, \mathrm{d}x = \left.\frac{e^{-sx}}{s}\right|_\infty^0 = \frac{1}{s} \quad (\mathrm{Re}\ s > 0)$$

The inverse is computed from the Bromwich integral

$$f(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)\, e^{sx}\, \mathrm{d}s = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{sx}}{s}\, \mathrm{d}s \quad (\alpha > 0)$$



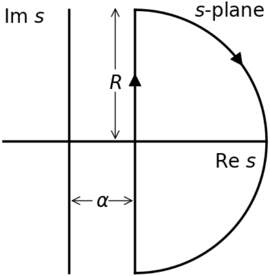
For $x > 0$, we take a contour $\mathcal{C}$ to the *left*

$$1 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{e^{sx}}{s}\, \mathrm{d}s$$

$$= \frac{1}{2\pi i}\left[\int_{\alpha-iR}^{\alpha+iR} \frac{e^{sx}}{s}\, \mathrm{d}s + \int_{\theta=\pi/2}^{3\pi/2} \underbrace{\frac{e^{(\alpha+Re^{i\theta})x}}{\alpha+Re^{i\theta}}\, iRe^{i\theta}}_{\sim\, e^{Rx\cos\theta}}\, \mathrm{d}\theta\right]$$

$$= f(x) \qquad \ldots R \to \infty$$

Continued



For $x < 0$, we take a contour $\mathcal{C}$ to the *right*

$$0 = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{e^{sx}}{s} \, \mathrm{d}s$$

$$= \frac{1}{2\pi i} \left[\int_{\alpha - iR}^{\alpha + iR} \frac{e^{sx}}{s} \, \mathrm{d}s + \int_{\theta=\pi/2}^{-\pi/2} \underbrace{\frac{e^{(\alpha + Re^{i\theta})x}}{\alpha + Re^{i\theta}} \, iRe^{i\theta}}_{\sim e^{Rx\cos\theta}} \, \mathrm{d}\theta \right]$$

$$= f(x) \qquad \ldots R \to \infty$$

Thus the Bromwich integral yields

$$\boxed{f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}}$$

Useful Transforms

Evaluate the Laplace transform of the following functions

$f(x) = \delta(x - x_0) \qquad (x_0 > 0)$

$$\mathcal{L}[f(x)] = \int_0^\infty e^{-sx}\, \delta(x - x_0)\, \mathrm{d}x = e^{-sx_0}$$

$f(x) = \sin \lambda x$

$$\begin{aligned} \mathcal{L}[f(x)] = \int_0^\infty e^{-sx} \sin \lambda x \, \mathrm{d}x &= \frac{1}{2i}\left[\frac{e^{x(i\lambda - s)}}{(i\lambda - s)} + \frac{e^{-x(i\lambda + s)}}{(i\lambda + s)}\right]_0^\infty \\ &= \frac{\lambda}{s^2 + \lambda^2} \qquad (\mathrm{Re}\ s > 0) \end{aligned}$$

$f(x) = \cos \lambda x$

$$\begin{aligned} \mathcal{L}[f(x)] = \int_0^\infty e^{-sx} \cos \lambda x \, \mathrm{d}x &= \frac{1}{2}\left[\frac{e^{x(i\lambda - s)}}{(i\lambda - s)} - \frac{e^{-x(i\lambda + s)}}{(i\lambda + s)}\right]_0^\infty \\ &= \frac{s}{s^2 + \lambda^2} \qquad (\mathrm{Re}\ s > 0) \end{aligned}$$

Useful Transforms

$$f(x) = x^n$$

$$\mathcal{L}[f(x)] = \int_0^\infty e^{-sx}\, x^n \,\mathrm{d}x = (-1)^n \frac{\mathrm{d}}{\mathrm{d}s^n} \int_0^\infty e^{-sx} \mathrm{d}x = (-1)^n \frac{\mathrm{d}}{\mathrm{d}s^n} \frac{1}{s}$$

$$= \frac{n!}{s^{n+1}} \quad (\mathrm{Re}\ s > 0)$$

$$f(x) = e^{-\lambda x}$$

$$\mathcal{L}[f(x)] = \int_0^\infty e^{-sx}\, e^{-\lambda x} \,\mathrm{d}x = \left. \frac{e^{-(\lambda+s)x}}{\lambda + s} \right|_\infty^0 = \frac{1}{s+\lambda} \quad (\mathrm{Re}\ (s+\lambda) > 0)$$

Table of Laplace Transforms

$f(x)$	$\mathcal{L}[f(x)] = F(s)$	Convergence Condition
1	$\frac{1}{s}$	Re $s > 0$
$\delta(x - x_0) \quad (x > x_0)$	e^{sx_0}	
$\sin \lambda x$	$\frac{\lambda}{s^2 + \lambda^2}$	Re $s > 0$
$\cos \lambda x$	$\frac{s}{s^2 + \lambda^2}$	Re $s > 0$
x^n	$\frac{n!}{s^{n+1}}$	Re $s > 0$
$e^{-\lambda x}$	$\frac{1}{s + \lambda}$	Re $(s + \lambda) > 0$