Lecture 29: Physical Applications

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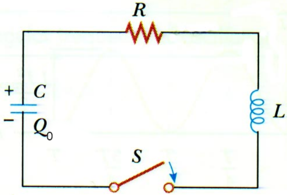
LCR circuit

Find the current in the circuit shown below if the switch is closed at time $t = 0$. You may take the initial charge on the capacitor as Q_0



$$RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C} = 0$$

Solution

$$\text{Since } \frac{\mathrm{d}Q}{\mathrm{d}t} = I \implies Q(t) = Q_0 + \int_0^t I(t')\,\mathrm{d}t'$$

$$RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}\left[Q_0 + \int_0^t I(t')\,\mathrm{d}t'\right] = 0$$

Take Laplace transform, and denote $\mathcal{L}[I(t)] = i(s)$

$$R\,i(s) + L\,(s\,i(s) - I(0)) + \frac{1}{C}\left[\frac{Q_0}{s} + \frac{i(s)}{s}\right] = 0$$

With $I(0) = 0$, we solve for $i(s)$

$$i(s) = -\frac{Q_0}{LC}\,\frac{1}{(s+a)^2 + b^2}$$

with

$$a = \frac{R}{2L} \qquad b = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

We obtain the current

$$\boxed{I(t) = \mathcal{L}^{-1}[i(s)] = -\frac{Q_0}{LC}\,\frac{e^{-at}\sin\,bt}{b}}$$

using either the table

$$\mathcal{L}[e^{-at}\sin\,bt] = \frac{b}{(s+a)^2 + b^2}$$

Bromwich integral with poles $-a \pm ib$

Signal Multiplication

Consider two input signals

$$
\begin{aligned}
u(t) &= \begin{cases} e^{-at} & (t > 0) \\ 0 & (t < 0) \end{cases} \\
v(t) &= \cos \omega_0 t
\end{aligned}
$$

sent to a multiplier which produces the output

$$w(t) = u(t)v(t)$$

Derive $\tilde{w}(\omega)$ and hence the output spectrum $|\tilde{w}(\omega)|^2$

Solution

Method 1: Brute force

$$
\begin{aligned}
\tilde{w}(\omega) = \int_0^\infty e^{-i\omega t}\, e^{-at} \cos \omega_0 t \, \mathrm{d}t = \int_0^\infty e^{-(i\omega + a)t} \left[\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right] \mathrm{d}t \\
= \frac{1}{2} \left[\frac{1}{i(\omega - \omega_0) + a} + \frac{1}{i(\omega + \omega_0) + a} \right]
\end{aligned}
$$

Method 2: Convolution (reveals a big picture!)

$$\tilde{u}(\omega) = \int_0^\infty e^{-i\omega t}\, e^{-at}\, \mathrm{d}t = \frac{1}{i\omega + a}$$

$$\tilde{v}(\omega) = \int_{-\infty}^\infty e^{-i\omega t} \cos \omega_0 t\, \mathrm{d}t = \pi \underbrace{[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]}_{\text{two Dirac impulses}}$$

By convolution theorem

$$\begin{aligned}\tilde{w}(\omega) &= \frac{1}{2\pi}\,(\tilde{v} * \tilde{u})(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{u}(\omega-\omega')\,\tilde{v}(\omega')\,\mathrm{d}\omega' \\ &= \frac{1}{2} \underbrace{\left[\frac{1}{i(\omega-\omega_0)+a} + \frac{1}{i(\omega+\omega_0)+a}\right]}_{\text{Dirac impulses replaced by copies of } \tilde{u} \text{ and shifted by } \pm\omega_0}\end{aligned}$$

Visualization

Visualize all this by plotting $|\tilde{u}(\omega)|$, $|\tilde{v}(\omega)|$ and $|\tilde{w}(\omega)|$