1 INTRODUCTION [1]

A magnet may be regarded as consisting of a set of magnetic dipoles residing on the vertices of a crystal lattice. We refer to the magnetic dipoles as spins. The spins are able to exchange energy through interactions between themselves, as well as between themselves and other degrees of freedom of the crystal lattice (e.g. via spin-orbit coupling). We assume that we are dealing with a system described by a single temperature $T$. At high temperatures and zero external field, the system is in the para-magnetic phase: following the time evolution of any spin would reveal that it points in all directions with equal frequency. Thus, no direction is singled out at any given time when considering all of the spins in the system and the net magnetic moment is zero.

Below a critical temperature, $T_c$, however, the spins tend to align along a particular direction in space, even in the absence of an external field. In this case, there is a net magnetization, $M(T)$, and the system is in the ferromagnetic phase. The onset of this behavior is a continuous phase transition: the magnetization rises continuously from zero as the temperature is reduced below $T_c$. The magnetization is zero above the transition and is non-zero below the transition temperature. A quantity which varies in this way is referred to as an order parameter. The question naturally arises as to why the system should order along any particular direction. In certain systems, the actual dipole interactions between the atoms on the lattice restrict the spins to point parallel or anti-parallel to one particular direction, which we shall take to be the $z$-axis. In these systems, known as Ising ferro-magnet, each spin cannot rotate through all possible orientations, but instead can only point along the $+z$ or $-z$ directions. The Ising ferro-magnet is therefore relatively simple to study.

The interaction energy between neighboring spins in an Ising ferro-magnet is lowest when neighboring spins point in the same direction. However, there is another class of systems, known as Ising anti-ferro- magnets, in which the sign of the interaction energy between neighboring spins is such that the energy is lowered when neighboring spins point in opposite directions.

2 PHASE TRANSITION IN 1D [1]

Phase transitions at $T = 0$ may or may not disappear for $T > 0$. In fact, what happens depends on the dimensionality of the system. We will now see, using heuristic arguments due to Landau and Peierls, that in $d = 1$ for $T > 0$ there is no long range order (i.e. no ferromagnetic state), whereas for $d = 2$, long range order can exist above $T = 0$, with a transition at $T_c > 0$ to a para-magnetic (i.e. disordered) state. The dimension above which a given transition occurs for $T > 0$ is often referred to as the lower critical dimension. The term “long range order” simply means a state in which the degrees of freedom (spins here) order over arbitrarily long distance, such as the ferromagnetic state. The para-magnetic state does not have long range order: two spins widely separated will not, on the average, point in the same direction, whereas they will do so in the ferromagnetic state. At zero temperature, there are two possible phases at zero field: all spins up and all spins down. Consider the spin up phase. As the temperature is raised above zero, each spin executes a sort
of Brownian motion by virtue of being in thermal equilibrium. In the thermodynamic limit, the uncorrelated flipping of a finite number of spins, \( n \), cannot destroy long range order:

\[
\lim_{N \to \infty} \frac{N - n}{N}
\]

The only fluctuations which can potentially destroy long range order are those involving a thermodynamically large number of spins. In other words, a non-zero fraction of spins, \( f \), must be reversed on average:

\[
\lim_{N \to \infty} \frac{N - fN}{N} \leq 1
\]

When spins are flipped, there are many ways that this can occur for a given value of the overall magnetization. Thus, we need to compute the entropy corresponding to a state with given magnetization. In the ordered state the entropy is zero if all the spins are aligned, so the free energy is just

\[ F = -NJ \]

Now consider a state with two domains.

![Figure 1: 1D state with domains](image)

The energy is then given by

\[ E_N = -(N - 1)J + J \]

\[ S_N = k_B \ln N \]

The free energy difference between states with spins up and a state with domains is

\[ \Delta F = 2J - k_B \ln N \]

In the limit of large \( N \), \( \Delta F \) tends to \(-\infty\). Thus, the long range order state is unstable towards thermal fluctuations for \( T > 0 \): the magnetization in zero external field for \( T > 0 \) is strictly zero, whereas for \( T = 0 \) the magnetization is either +1 or -1.

![Figure 2: Phase diagram in Ising model at \( d = 2 \)](image)
3 PHASE TRANSITION IN 2D [1]

We consider a domain of flipped spins, in a background of spins with long range order, but now the domain is two-dimensional. Suppose the border between the flipped spins and the up spins contains \( n \) bonds. Then the energy difference between the state with a domain and one with complete long range order is \( \Delta E \sim 3Jn \).

To obtain entropy choose a point on the boundary. If the coordination number of the lattice is \( z \), then an upper bound on the number of configurations of the boundary or domain wall is \( z^n \). The precise number is less than this, because the boundary of just a single domain cannot intersect itself, by definition (otherwise there are two domains!). As a crude first guess, we could argue that at each step, the domain wall can only go in \( z - 1 \) directions, because we should disallow the step that would retrace the previous one. This still allows the boundary to intersect itself, so our estimate of the entropy will be an overestimate. Then, as in \( d = 1 \), we can compute the entropy difference due to the presence of the domain: \( \Delta S \sim k_B n \ln (z - 1) \).

Thus, the change in free energy due to a domain whose boundary contains \( n \) bonds is

\[
\Delta F_n = [2J - k_B T \ln (z - 1)]n. 
\]

Thus there exists a critical temperature \( T_c \) such that as

\[
T \to T_c = \frac{2J}{k_B \ln (z - 1)}
\]

then \( \Delta F \to \infty \) as \( n \to \infty \) and the system is unstable towards the formation of domains. Accordingly we anticipate a disordered, para-magnetic phase with \( M = 0 \). For \( 0 < T < T_c \), however, \( \Delta F \) is minimized by \( n \to 0 \), and the state with long range order is stable. Thus, for \( 0 < T < T_c \), the system exhibits a net magnetization \( M \), which can be either positive or negative, in the absence of an applied field. This magnetization is often referred to as spontaneous magnetization.

4 SPONTANEOUS SYMMETRY BREAKING [2]

As external field \( |H| \to 0 \), magnetization is given by

\[
M = \frac{\partial F}{\partial H} = \begin{cases} 
M_s, & \text{if } H > 0, \\
-M_s, & \text{if } H < 0.
\end{cases}
\]  

The Hamiltonian has degenerate ground states (more than one possible state with the same energy). But with the application of external field \( H \), the system picks one of the ground states. An Ising ferro-magnet has zero magnetization at high temperature, but below \( T_c \) it breaks its rotational symmetry and magnetizes along a particular direction which is decided by the external field.

![Figure 3: Magnetization in the presence of external field when (a) \( T > T_c \) and (b) \( T < T_c \).](attachment:image.png)
From the figures below we understand that at $T > T_c$ there is no spontaneous magnetization as the field vanishes. However, below $T_c$ there is spontaneous magnetization whose value depends on how the external field disappears!

References

[1] Nigel Goldenfeld Lectures on phase transition and the re-normalization group.
[2] Alex Coeytaux Symmetry breaking and Ising ferromagnet