Moment Generating function: $p(k)$

Also known as the characteristic function. Mathematically it is defined as the Fourier transform of $p(x)$

$$\therefore p(k) = \int_{-\infty}^{+\infty} dx\, e^{-ikx} p(x) = \langle e^{-ikx} \rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n \langle x^n \rangle}{n!}$$

——— (1)

It is easy to see that $p(k)$ is the generator of moments. Since,

$$\langle x^m \rangle = \frac{\partial}{\partial(-ik)^m} p(k) \Big|_{k=0} \quad \text{——— (2)}$$

Cumulant generating function: $\ln p(k)$

Also known as the logarithm of characteristic function

We write $\ln p(k)$ as a Mclaurin series in powers of $(-ik)$. Thus

$$\ln p(k) = F(k) = F(0) + F'(0)(-ik) + F''(0)\frac{(-ik)^2}{2!} + F'''(0)\frac{(-ik)^3}{3!} + \dots$$

——— (3)

where $F(0) = \ln p(0) = \ln(1) = 0$

& $F'(0) = \langle x \rangle_c$

$F''(0) = \langle x^2 \rangle_c$

$F'''(0) = \langle x^3 \rangle_c$

$\vdots$

$F^n(0) = \langle x^n \rangle_c$

are the coefficients that are realized as cumulants. $\therefore \ln p(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n \langle x^n \rangle_c}{n!}$ — (4)

It is thus easy to see why $\ln p(k)$ is a generator of cumulants. The m^{th} cumulant is straightforwardly written from eq^n (4)

$$\langle x^m \rangle_c = \frac{\partial}{\partial(-ik)^m} \ln p(k) \Big|_{k=0} \quad \text{— (5)}$$

Relation between moments and cumulants:

We start by taking derivatives of $\ln \tilde{p}(k)$ – the cumulant generating function.

1^{st} derivative:

$$\frac{\partial}{\partial(-ik)} \ln \tilde{p}(k) = \frac{1}{\tilde{p}(k)} \frac{\partial \tilde{p}(k)}{\partial(-ik)}$$

Taking $k \to 0$ and using ② and ③,

$$\langle x \rangle_c = \langle x \rangle$$

"Mean"

II^{nd} derivative:

$$\frac{\partial^2}{\partial(-ik)^2} \ln \tilde{p}(k) = \frac{-1}{\tilde{p}(k)^2} \left(\frac{\partial}{\partial(-ik)} \tilde{p}(k) \right)^2 + \frac{1}{\tilde{p}(k)} \frac{\partial^2 \tilde{p}(k)}{\partial(-ik)^2}$$

Taking $k \to 0$ and using ② and ③

$$\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$$

"Variance"

IIIrd derivative:

$$\frac{\partial^3}{\partial(-ik)^3} \ln p(k) = \frac{2}{p(k)^3}\left(\frac{\partial}{\partial(-ik)} p(k)\right)^3 - \frac{2}{p(k)^2}\frac{\partial p(k)}{\partial(-ik)}\frac{\partial^2 p(k)}{\partial(-ik)^2}$$

$$-\frac{1}{p(k)^2}\frac{\partial p(k)}{\partial(-ik)}\frac{\partial^2 p(k)}{\partial(-ik)^2} + \frac{1}{p(k)}\frac{\partial^3 p(k)}{\partial(-ik)^3}$$

Again taking $k \to 0$ and using definitions (2) and (5)

$$\langle x^3 \rangle_c = \langle x^3 \rangle - 3\langle x \rangle \langle x^2 \rangle + 2\langle x \rangle^3$$ "Skewness"

IVth derivative is left as an exercise but it is easy to show that

$$\langle x^4 \rangle_c = \langle x^4 \rangle - 4\langle x^3 \rangle \langle x \rangle - 3\langle x^2 \rangle^2 + 12\langle x^2 \rangle \langle x \rangle^2 - 6\langle x \rangle^4$$

"Kurtosis"

These cumulants provide a useful and compact way to describe a PDF.

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Graphical method to generate moments:

The expressions for mean, variance, skewness and kurtosis can be inverted to write expressions for moments. Below we present a much more powerful method for generating moments in terms of cumulants.

Generate the m^{th} moment by summing over all possible subdivisions of m points into groupings of smaller (connected or disconnected) clusters. The connected clusters are cumulants and the disconnected are moments.

$\langle x \rangle =$

$\langle x \rangle_c$

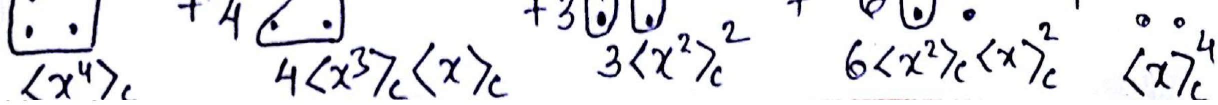
$\langle x^2 \rangle =$

$\langle x^2 \rangle_c \quad \langle x \rangle_c^2$

$\langle x^3 \rangle =$

$\langle x^3 \rangle_c \quad 3\langle x^2 \rangle_c \langle x \rangle_c \quad \langle x \rangle_c^3$

$\langle x^4 \rangle =$



$\langle x^4 \rangle_c \quad 4\langle x^3 \rangle_c \langle x \rangle_c \quad 3\langle x^2 \rangle_c^2 \quad 6\langle x^2 \rangle_c \langle x \rangle_c^2 \quad \langle x \rangle_c^4$

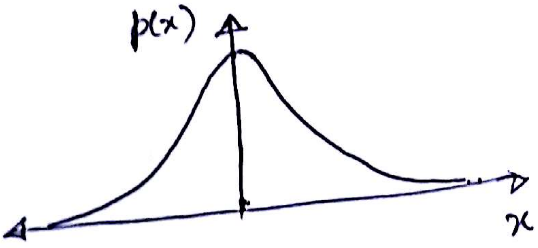
Physical Significance of cumulants in natural world

In statistical physics, many extensive quantities (size dependent) are related to the cumulants. A deep connection exists between thermodynamic observables and cumulants of some extensive quantities. Below, I give some examples.

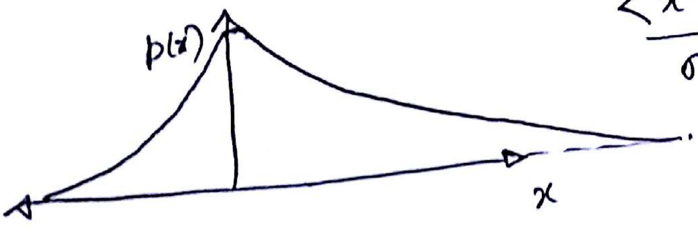
IInd cumulant or variance: Measures the spread of distribution around its mean.

For eg. Heat capacity : $C_V = \frac{1}{k_B T^2} \langle E^2 \rangle_c = \frac{1}{k_B T} \left(\langle E^2 \rangle - \langle E \rangle^2 \right)$

IIIrd cumulant or skewness: Its a measure of asymmetry in the PDF. For eg.



$\frac{\langle x^3 \rangle_c}{\sigma^3} = 0$: No skew. The PDF is symmetric



$\frac{\langle x^3 \rangle_c}{\sigma^3} > 0$: Positive skew. The PDF has more points towards positive tail.

$\frac{\langle x^3 \rangle_c}{\sigma^3} < 0$: Negative skew. The PDF has more points towards negative tail.