Conserved Charges in asymptotically deSitter spacetimes

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Outline

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- Existence of de Sitter solutions in String theory are still widely debated.
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- Existence of de Sitter solutions in String theory are still widely debated.
- Even from a purely gravitational perspective, many issues are not understood e.g. gravitational waves in de Sitter, appropriate boundary conditions for "de Sitter holography".
- Our work aims to compare different notions of gravitational charges.

deSitter Spacetime

- de Sitter spacetime is defined as the maximally symmetric solution to Einstein's equation with a positive cosmological constant ($\Lambda > 0$).

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0.$$

- In global coordinates, the metric is

$$ds^2 = -d\tau^2 + (\cosh\tau)^2 d\sigma_{d-1}^2,$$

where $\tau \in (-\infty, \infty)$ and $d\sigma_{d-1}^2$ is the round metric on $(d-1)$ unit sphere.

- In the Poincare/Cosmological patch, the metric is

$$ds^2 = -dt^2 + e^{2t}\delta_{ij} dx^i dx^j,$$

where $t \in (-\infty, \infty)$ and everywhere, we have taken the de Sitter radius $\ell = 1$.

de Sitter spacetime

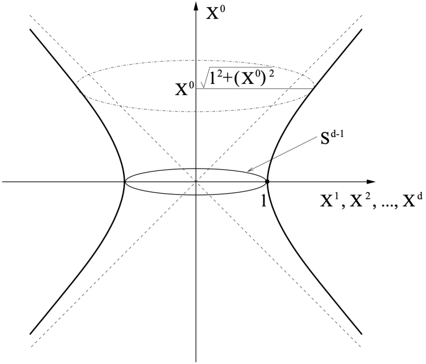


Figure: Global de Sitter viewed as an embedding in $R^{1,d}$

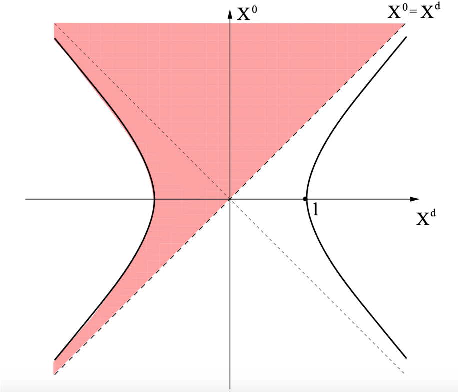


Figure: Poincare patch of de Sitter is the shaded region

de Sitter Conformal

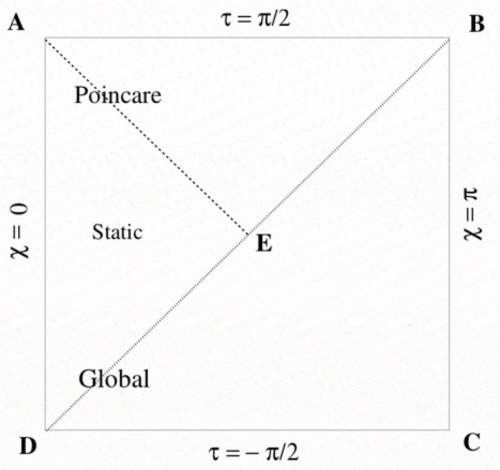


Figure: The conformal diagram of de Sitter spacetime

Conserved charges in general relativity

- In other theories of physics, once we have a conserved energy momentum tensor we can define conserved charges.
- As we all know, there is no *local* notion of energy-momentum for gravity in GR - equivalence principle.
- Defining conserved charges are therefore tricky and they involve global notions.
- Familiar examples might be the Komar integrals or the ADM mass in the asymptotically flat case and using the Balasubramaniam Krauss boundary stress tensor in asymptotically AdS

- There exist many similar definitions in de Sitter literature.
- We considered 3 conceptually and quantitatively very different such definitions and tried to compare them.
- We call them the ABK, MK and counterterm charges.
- We can derive (and therefore interpret) the ABK charges using a Hamiltonian formalism. And we show that it is equivalent to MK charges and upto a constant offset, equivalent to the counterterm charges as well.
- Before we do that, we need to define our phase space and then use the covariant phase space formalism to derive the charges.

Future Asymptotically de Sitter spacetimes

- We have a conformal compactification $\widetilde{\mathcal{M}} = \mathcal{M} \cup \mathcal{I}^+$, $\widetilde{g}_{ab} = \Omega^2 g_{ab}$ with $\Omega = 0$ on $\mathcal{I}^+$ and $\nabla_a \Omega$ is nowhere vanishing on $\mathcal{I}^+$.
- The induced metric on $\mathcal{I}^+$ is locally isometric to the round metric on $(d-1)$ unit sphere $\mathbb{S}^{d-1}$.
- It is possible to choose the conformal factor in such a way that the unphysical metric in a neighbourhood of $\mathcal{I}^+$ takes the form

$$\tilde{g}_{ab} = -\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + \tilde{h}_{ab}(\Omega).$$

- For example, in this choice, the unphysical metric for global de Sitter will look like,

$$\tilde{h}_{ab}(\Omega) = \left(1 + \frac{1}{2}\Omega^2 + \frac{1}{16}\Omega^4\right)(\tilde{h}_{ab})_0,$$

where $(\tilde{h}_{ab})_0$ is the round metric on the unit $(d-1)$-sphere. This is achieved by setting $\tau = -\ln(\frac{\Omega}{2})$ and then conformally rescaling the whole metric.

- Using an ADM type decomposition and an asymptotic expansion in Ω

$$\tilde{h}_{ab} = \Sigma_{i=0}^{\infty} (\tilde{h}_{ab})_i \Omega^i,$$

we can try to solve Einstein's equations recursively.

- One can then write down the metric of a general asymptotically dS spacetime in the following way

$$\tilde{g}_{ab} = \begin{cases} -\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + \left(1 + \frac{1}{2}\Omega^2\right) (\tilde{h}_{ab})_0 - \frac{2}{3}\Omega^3 \tilde{E}_{ab} + \mathcal{O}(\Omega^4) & d = 4, \\ -\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + \left(1 + \frac{1}{2}\Omega^2 + \frac{1}{16}\Omega^4\right) (\tilde{h}_{ab})_0 - \frac{1}{2}\Omega^4 \tilde{E}_{ab} + \mathcal{O}(\Omega^5) & d = 5, \\ -\tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega + \left(1 + \frac{1}{2}\Omega^2 + \frac{1}{16}\Omega^4\right) (\tilde{h}_{ab})_0 - \frac{2}{d-1}\Omega^{d-1} \tilde{E}_{ab} + \mathcal{O}(\Omega^d) & d \geq 6, \end{cases}$$

- Where we define the electric part of the Weyl tensor as,

$$\tilde{E}_{ac} = \frac{1}{d-3}\Omega^{3-d}\left(\tilde{C}_{abcd}\tilde{n}^b\tilde{n}^d\right),$$

- Expanding $\tilde{E}_{ab}$ like above, one can show

$$(\tilde{h}_{ab})_{d-1} = -\frac{2}{d-1}(\tilde{E}_{ab})_0.$$

- $\tilde{E}_{ab}$ contains all the information about the complete spacetime and therefore characterises a *particular* solution in the phase space.

Covariant Phase space - Construction

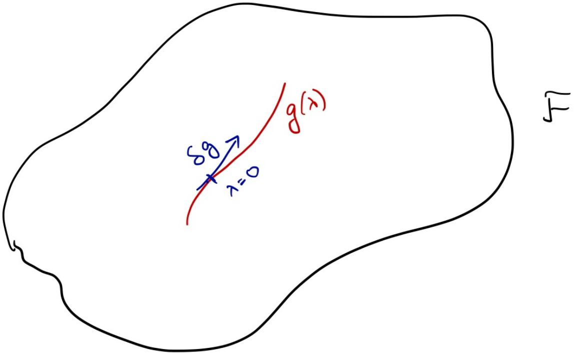
- We use the covariant phase space framework to construct our notion of a conserved charge. [Crnkovic,Witten; Lee,Wald; Ashtekar,Bombelli,Ruela; ...]
- This is nothing but the space of all solutions of the theory.
- The Lagrangian is a function on this space and its variation is given by,

$$\delta L = E(g)_{ab}\delta g^{ab} + d\theta(g, \delta g),$$

  where θ is called the presymplectic potential $(d-1, 1)$-form. This is the central object in this formalism.
- We define the presymplectic current $(d-1, 2)$-form ω as,

$$\omega(g, \delta_1 g, \delta_2 g) = \delta_1\theta(g, \delta_2 g) - \delta_2\theta(g, \delta_1 g).$$

Figure: $F :=$ Space of all metrics. δg is thought of as a tangent vector on F



Charge

- Integrating presymplectic current over a Cauchy hypersurface[1] we obtain a presymplectic form,

$$\Omega(g,\delta_1 g,\delta_2 g) = \int_\Sigma \omega(g,\delta_1 g,\delta_2 g).$$

- Now, a vector field ξ on space-time manifold M with metric g (a point on $\bar{\mathcal{F}}$) naturally induces the field variation $\delta_\xi g = \mathcal{L}_\xi g$, which is a tangent on $\bar{\mathcal{F}}$.
- The Hamiltonian function H_ξ conjugate to ξ is defined to be

$$\begin{aligned}\delta H_\xi = \Omega(g,\delta g,\mathcal{L}_\xi g) &= \int_\Sigma \omega(g,\delta g,\mathcal{L}_\xi g)\\ &= \int_{\partial\Sigma} \left[\delta Q_\xi - \xi\cdot\theta\right].\end{aligned}$$

- The above equation does not ensure the existence of H_ξ.
- A necessary and sufficient condition for H_ξ to exist is given by,

$$\int_{\partial\Sigma} \xi\cdot\omega(g,\delta_1 g,\delta_2 g) = 0.$$

[1] The integrals are defined with appropriate boundary conditions for fields to ensure that the

Charge for asymptotically dS

- In our current problem, $L_{a_1 \ldots a_d} = \frac{1}{16\pi G}(R - 2\Lambda)\epsilon_{a_1 \ldots a_d}$
- The most general variation consistent with our gauge choice and boundary condition is of the form,

$$\delta g_{ab} = -\frac{2}{d-1}\Omega^{d-3}\left(\delta \tilde{E}_{ab} + \mathcal{L}_\eta \tilde{E}_{ab}\right) + \mathcal{O}(\Omega^{d-2})$$

  where η is an arbitrary diffeomorphism with

$$\mathcal{L}_\eta \bar{g} = \mathcal{O}(\Omega^{d-2}),$$

  and where $\bar{g}$ is the background de Sitter metric.
- One can then show that both θ and ω vanish on $\mathcal{I}^+$.
- From conservation properties of the presymplectic current, it follows that the *presymplectic form* $\Omega(g, \delta_1 g, \delta_2 g)$ *vanishes on all complete Cauchy slices.*

- We can still define (at least formally) a useful notion of conserved charges if we choose spacelike hypersurfaces Σ in M that smoothly cuts $\mathcal{I}^+$ as shown below,

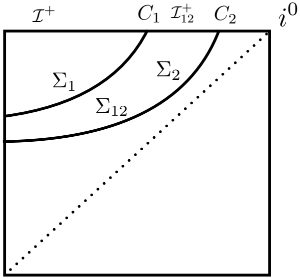


Figure: Hypersurfaces Σ_1 and Σ_2 together with the portion $\mathcal{I}^+_{12}$ of $\mathcal{I}^+$ enclosing the spacetime volume Σ_{12}.

- Here $C_i \subset \mathcal{I}^+$ are codimension 2 surfaces.

Hamiltonian charges for asymptotically dS

- Then from the fact that the presymplectic potential and current vanishes on $\mathcal{I}^+$, it follows that H_ξ exists, is independent of hypersurface approaching C and is given by

$$\delta H_\xi = \int_C \delta Q_\xi .$$

- For our class of metric variations we have,

$$(\delta Q_\xi)_{a_1 \ldots a_{d-2}} = \frac{1}{8\pi G} \tilde{\epsilon}_{a_1 \ldots a_{d-2} bc} (\tilde{\nabla}^b \Omega) \delta \tilde{E}^c{}_d \xi^d + \mathcal{O}(\Omega).$$

- Taking the reference spacetime to be pure de Sitter, we have the following nice result

$$\boxed{H_\xi[C] := \frac{1}{8\pi G} \int_C \tilde{E}_{ab} \tilde{u}^b \xi^a \widetilde{dS},}$$

where $\tilde{u}^a$ is defined thus ${}^{(d)}\tilde{\epsilon} = \tilde{u} \wedge \tilde{n} \wedge {}^{(d-2)}\tilde{\epsilon}$.

ABK Charges

- Ashtekar, Bonga and Kesavan [arXiv:1409.3816], proposed the following definition of charge

$$Q_{\xi}^{ABK}[S] := -\frac{1}{8\pi G}\oint_{S} E_{ab}u^{a}\xi^{b}dS,$$

  where $E_{ab} = C_{acbd}n^{c}n^{d}$ is the electric part of the Weyl tensor, u^{a} is the outward pointing normal to the S^{d-2} within Σ.
- The above definition was proposed after analysing asymptotic equations of motion and observing that the electric part of the Weyl tensor E_{ab} is traceless and conserved at $\mathcal{I}^{+}$. It then follows that $E_{ab}\xi^{a}$ is a conserved current.

ABK Charges

- We are able to reproduce the exact same expression ABK have, and at least formally, we are able to interpret the charge as a hamiltonian that generates a diffeomorphism on phase space.
- We are also able to say why Electric part of the Weyl tensor is entering the charge formula - it is the tensor one must provide over and above our definition of asymptotically dS spacetime to fully specify the spacetime.

Counter-term charges

- Inspired by AdS/CFT methods, there is another way of defining charges called the counter term method [Strominger [2001]; Balasubramanian, de Boer, Minic [2002]; ...].
- Charges associated to asymptotic symmetry ξ^a are constructed from a boundary stress tensor, which is obtained by varying the effective boundary Lagrangian,

$$Q_\xi^{\mathrm{ct}}[C] = \lim_{C_\Omega \to C} \int_{C_\Omega} \tau_{ab}\, \xi^a\, u^b\, dS,$$

  where τ_{ab} is the boundary stress tensor. This discussion is slightly more general. It is for asymptotically locally de Sitter.
- In $d = 5$

$$\tau_{ab} = \frac{1}{8\pi G}\left[(Kh_{ab} - K_{ab}) + (d-2)h_{ab} + \frac{1}{d-3}\left(\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}h_{ab}\right)\right].$$

Counter-term charges

- In five dimensions, the difference between counterterm charge and the ABK charge can be written as

$$Q_\xi^{\mathrm{ct}}[C] - H_\xi[C] = \frac{1}{16\pi G}\int_C \tilde{\Delta}_{ab}\tilde{u}^b\xi^a\widetilde{dS},$$

where

$$\tilde{\Delta}_{ab} = -\frac{1}{4}\left(\frac{2}{3}\tilde{\mathcal{R}}\tilde{\mathcal{R}}_{ab} - \frac{1}{4}\tilde{\mathcal{R}}^2\tilde{h}_{ab} - \tilde{\mathcal{R}}_a^c\tilde{\mathcal{R}}_{cb} + \frac{1}{2}\tilde{\mathcal{R}}_{cd}\tilde{\mathcal{R}}^{cd}\tilde{h}_{ab}\right).$$

- It is clear from the above that the difference between the two charges is given purely by the curvature of the boundary metric and is therefore independent of the specific asymptotic de Sitter spacetime under consideration.

Trace anomaly

- The difference $\tilde{\Delta}_{ab}$ can also be compared with the trace anomaly computed in[Nojiri, Odinstov [2001]]. The trace of $\tilde{\Delta}_{ab}$ is simply the trace of τ_{ab}. We have

$$\tau_a{}^a = -\frac{1}{64\pi G}\left(\tilde{\mathcal{R}}_{ab}\tilde{\mathcal{R}}^{ab} - \frac{1}{3}\tilde{\mathcal{R}}^2\right).$$

- This answer is just the negative of the AdS result.

MK Charges

- During the construction of our phase space, we mentioned that the symplectic structure on any cauchy hypersurface necessarily vanishes since we keep the metric fixed at $\mathcal{I}^+$.
- This motivated Marolf and Kelly [arXiv:1202.5347] to define their phase space differently.
- They proposed a definition of asymptotically de Sitter spacetimes (with non-compact Cauchy slices) with appropriate fall-off near spatial infinity i^0 without reference to $\mathcal{I}$. In this sense, their set-up is conceptually very different from ABK.
- Nonetheless, there exist spacetimes that satisfy both boundary conditions - for example, Schwarzschild & Kerr de Sitter spacetime. Therefore one is tempted to compare both these charges.

MK Charges

- In this case, the expression for charges is,

$$Q_\xi^{\mathrm{KM}}[C] = \lim_{r\to\infty} \frac{1}{8\pi G} \int_{(\partial\Sigma)_r} \Delta' \pi_{ab}\, \xi^a u^b dS.$$

  where

$$\Delta'\pi^{ab} = \pi^{ab} - \bar{\pi}^{ab} = \pi^{ab} + (d-2)h^{ab},$$

  Σ is a slice that approaches i^0, u^a is the unit normal to $(\partial\Sigma)_r$ in Σ.
- With the Marolf Kelly fall-offs,

$$\Delta h_{ab} = \mathcal{O}(r^{-(d-1)}), \quad \Delta K_{ab} = \mathcal{O}(r^{-(d-2)}), \quad \Delta K = \mathcal{O}(r^{-(d-2)}).$$

  One can show that,

$$Q_\xi^{\mathrm{KM}}[C] = H_\xi[C]$$

  because

$$(d-3)\Delta'\pi_{ab} - (C_{acbd} n^c n^d) \to 0 \text{ as } r \to \infty.$$

- Therefore, the charges are the same at spatial infinity for the class of spacetimes that satisfy both Kelly-Marolf and ABK boundary conditions.

Summary

- We defined asymptotically de Sitter spacetimes and wrote down a Fefferman - Graham type expansion near $\mathcal{I}^+$.
- We defined our phase space of asymptotically de Sitter spacetimes to be the space of all metrics that can be put in Fefferman - Graham form and saw that $\tilde{E}_{ab}$ completely characterises an element of the phase space.
- We reproduced ABK formula for the conserved charge using the covariant phase space approach in our phase space.
- We compared it with the counter term method of obtaining charges and see that they differ only by terms corresponding to the boundary metric, which is non-dynamical.
- We also compare ABK charge with Marolf Kelly charge and see that they match exactly.

Outlook

1. There are various possible directions one could explore.
2. There are several other notions of asymptotically de Sitter spacetimes, e.g., [Traschen, Kastor [2002]] and [Strominger et al [2010, 2011]]. These are again conceptually different from ABK. It will be good to understand connection to this literature.
3. In recent literature Bondi coordinates are also explored for de Sitter. It will be useful to relate Bondi expansion to our F-G expansion.
4. Exploring the consequences of allowing matter to enter our discussion.
5. Allowing for gravitational radiation to leak away to infinity, yet managing to reduce the asymptotic symmetry group at $\mathcal{I}^+$ from full $\mathit{diff}(\mathcal{I}^+)$

Thank You

and please do send me any feedback

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