Nonlinear Langevin dynamics via holography

Bidisha Chakrabarty

ICTS-TIFR, Bangalore

based on

arXiv: 1906.07762

with Joydeep Chakravarty, Soumyadeep Chaudhuri, Chandan Jana, R. Loganayagam, Akhil Sivakumar

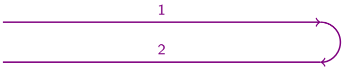
CSGC 2020

Plan of the talk

Table of Contents

Introduction

- Non-equilibrium processes are ubiquitous in nature.
- Useful to develop a framework to study such processes.
- Path integral techniques in QFT are tailored to study equilibrium states.
- They have to be extended to analyse non-equilibrium cases.
- Such an extension is given by the Schwinger-Keldysh (SK) path integral formalism on a folded time-contour.



Introduction

- Nevertheless a direct computation of observables from microscopic theory may be complicated.
- Depending on the observables of interest, one chooses appropriate d.o.f to define an open quantum system.
- The system acts as a probe, coupled to an environment.
- System's dynamics is governed by an effective theory.
- The form of the effective theory is determined using the unitarity of the microscopic theory and its underlying symmetries.

Introduction

- For open systems, the effective theory mostly developed has been quadratic.
- The simplest example is provided by a Brownian particle weakly interacting with a large thermal bath.
- Effective theories for such a Brownian particle (probe) were derived by Feynman-Vernon, Caldeira-Leggett and others using SK path integral formalism.
- The quadratic effective theory is used to compute two-point correlators of the system (linear response).

Introduction

- The effective dynamics has a classical stochastic description given by linear Langevin equation with a Gaussian noise.

$$\frac{d^2q}{dt^2} + \gamma \frac{dq}{dt} = \langle f^2 \rangle \mathcal{N} \; . \tag{1}$$

- The probability distribution of noise ensemble :

$$P[\mathcal{N}] \propto \exp\left\{ -\frac{\langle f^2 \rangle}{2} \int dt \; \mathcal{N}^2 \right\} \; . \tag{2}$$

- Variance of noise $\langle f^2 \rangle$ and damping γ are related by Fluctuation-Dissipation relation (FDR)

$$\langle f^2 \rangle = \frac{2}{\beta}\gamma. \tag{3}$$

- The path integral in the stochastic problem is related to an underlying quantum path integral.

Table of Contents

Nonlinear Langevin equation

- Any real system exhibits nonlinearities in its dynamics and non-Gaussianities in noise.
- Such a system is better described by a non-linear Langevin equation with non-Gaussian noise.
- This equation is given by

$$\begin{aligned}\mathcal{E}[q,\mathcal{N}] \equiv \ddot{q} + \left(\gamma + \zeta_\gamma \mathcal{N}^2\right)\dot{q} + \left(\overline{\mu}^2 + \zeta_\mu \mathcal{N}^2\right) q + \mathcal{N}\left(\overline{\zeta}_3 - \overline{\zeta}_{3\gamma}\frac{d}{dt}\right)\frac{q^2}{2!} \\ + \left(\overline{\lambda}_4 - \overline{\lambda}_{4\gamma}\frac{d}{dt}\right)\frac{q^3}{3!} = \langle f^2\rangle \mathcal{N} \\ \mathcal{N} \rightarrow \text{thermal noise} \\ \zeta_\gamma \rightarrow \text{thermal jitter in damping constant} \\ \zeta_\mu \rightarrow \text{jitter in renormalised frequency} \\ \overline{\zeta}_3, \overline{\zeta}_{3\gamma}, \overline{\lambda}_4, \overline{\lambda}_{4\gamma} \rightarrow \text{anharmonicity parameters} \; .\end{aligned} \tag{4}$$

Non-linear Langevin equation

- The probability distribution of the noise is

$$P[\mathcal{N}] \sim \exp\Big[-\int dt\Big(\frac{\langle f^2\rangle}{2}\mathcal{N}^2 + \frac{Z_I}{2}\dot{\mathcal{N}}^2 + \frac{\zeta_N}{4!}\mathcal{N}^4\Big)\Big] \tag{5}$$

$\zeta_N \to$ non-Gaussianity of the thermal noise .

- Generalised fluctuation-dissipation relation for a Brownian particle is

$$\zeta_N = -\frac{12}{\beta}\zeta_\gamma \ . \tag{6}$$

- This is a consequence of the microscopic time reversal invariance and thermality of the bath.

Heavy quark in SK formalism

- Consider a heavy quark in a strongly coupled CFT_d plasma at temperature T.
- The quark behaves as a Brownian particle moving in a d-dimensional spacetime with spatial position q^i $(i = 1, 2, \ldots, d-1)$.
- SK effective Lagrangian of a Brownian particle weakly coupled to a thermal bath till quartic order has been calculated.

Heavy quark in SK formalism

- Need two copies of each d.o.f on two legs of SK contour for evolution of ket/ bra of density matrix.
- Integrating out bath introduces corrections to particle's SK action, referred to as 'influence phase'.
- In general, influence phase is non-local in time.
- One can expand the influence phase of the particle in a derivative expansion.

Heavy quark in SK formalism

- In a strongly coupled CFT, direct computation of quark's influence phase is nearly impossible.
- General structure of local effective action can be written using microscopic unitarity of (quark+bath) combined system, hermiticity of operators and invariance of microscopic dynamics under constant translations and rotations of q^i's.
- The local effective action of quark till quartic order is

$$L_{SK} \equiv m_p \frac{dq_a^i}{dt}\frac{dq_d^i}{dt} - m_p\gamma \frac{dq_a^i}{dt} q_d^i + i m_p^2 \frac{\langle f^2 \rangle}{2!} q_d^2 - i m_p^2 \frac{Z_I}{2!}\left(\frac{dq_d}{dt}\right)^2 + m_p^3 \zeta_\gamma q_d^2 \frac{dq_a^i}{dt} q_d^i + i m_p^4 \frac{\zeta_N}{4!}(q_d^2)^2 , \tag{7}$$

where $q_a = \frac{1}{2}(q_R + q_L)$, $q_d = q_R - q_L$; q_R and q_L being the right and left d.o.f of SK contour.

Table of Contents

Outline of our work

- We compute non-local influence phase of quark in a strongly coupled CFT using holographic SK path integral framework.
- We verify the generalised fluctuation-dissipation relations using holography.

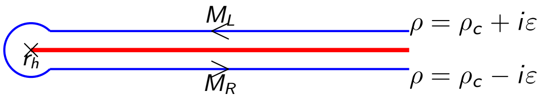
Table of Contents

Holographic-SK prescription

- A general prescription to compute realtime n-point functions defined on a SchwingerKeldysh contour using gravity. Skenderis, van Rees
- The holographic prescription amounts to filling in this contour with bulk solutions: real segments of the contour are filled in with Lorentzian solutions while imaginary segments are filled in with Euclidean solutions.
- Appropriate matching conditions (roughly, the bulk fields and their derivatives should be continuous) are imposed at the corners of the contour.
- With the full path integration contour on the gravity side, the generating functional can be obtained by integrating over the bulk fields with sources as boundary conditions.

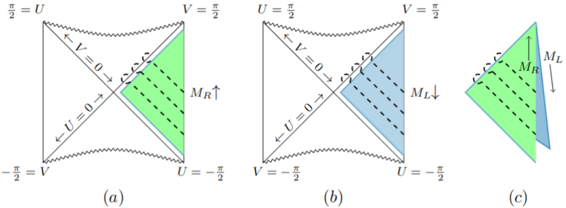
Holographic-SK prescription

- Consider AdS$_{d+1}$ black-brane geometry.
- Double black-brane spacetime to construct a configuration with CFT SK contour in boundary.
- Crossley-Glorioso-Liu have given a prescription recently clarifying the near horizon structure of this doubled black-brane space-time.
- Geometry along the radial direction: M_L, M_R are doubled manifolds and are stitched by a horizon cap.



- Geometry along the time direction: AdS boundaries asymptote to CFT SK contour.

Holographic-SK prescription



- In AdS/CFT, a heavy quark at boundary is dual to an open string in the bulk hanging from AdS boundary probing a black-brane geometry.
- The string stretches from two AdS boundaries and loops around a region obtained by radial Wick rotation connecting two stretched horizons.
- Horizon cap region regulates outgoing string modes that otherwise blow up at horizon.

The steps

- Expand Nambu-Goto action of the string till quartic order in fluctuations.
- Solve linearised string e.o.m with appropriate boundary conditions.
- Calculate quartic order onshell action with this.
- Integrate out radial coordinate to get boundary effective action of the particle.

Nambu-Goto action

- The black-hole metric in AdS_{d+1},

$$ds^2 = 2\ dvdr - r^2 \left(1 - \frac{r_h^d}{r^d}\right) dv^2 + r^2 dx_{d-1}^2\ .$$

- The inverse Hawking temperature $\beta = \frac{4\pi}{dr_h}$.
- Evaluating Nambu-Goto action on the double string configuration gives quark's influence phase.
- The string action is given by

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h(X)}\,,$$

- $h(X)$ is the determinant of induced metric on the string world-sheet. We choose

$$\sigma^\mu = (v, r)\,, \qquad X^i = X^i(v, r)\ \text{ for } i \in 1, ..., d-1$$

Nambu-Goto action

- Define new variables.

$$\rho \equiv \frac{r}{r_h}\,, \qquad \frac{d\xi}{d\rho} \equiv \frac{d}{2\pi i}\frac{\rho^{d-4}}{\rho^d - 1}\,, \qquad \lambda \equiv \frac{16\pi}{\alpha'}\,, \qquad \eta \equiv \frac{r_h^3 d}{2\pi i}.$$

- ξ is a *negative imaginary* radial co-ordinate in black-brane exterior.
- ξ resembles worldsheet tortoise co-ordinate $\xi \equiv \frac{d}{2\pi i}\oint_{\rho_c+i\epsilon}^{\rho} \frac{y^{d-4}dy}{y^d-1}$.
- ξ has a branch cut: $\xi(\rho_c + i\epsilon) = 0, \xi(\rho_c - i\epsilon) = 1$.

Nambu-Goto action

- The action becomes

$$S_{NG} = -\frac{r_h}{2\pi\alpha'}\int dv \oint d\rho \sqrt{1 + \left(\frac{\eta}{r_h}\frac{d\xi}{d\rho}\right)\left(\frac{\partial X^i}{\partial \xi} + i\beta\rho^2\frac{\partial X^i}{\partial v}\right)\frac{\partial X^i}{\partial \xi} + \ldots}$$

- Expand the action in fluctuations upto quartic order.

$$S_{NG} \approx S^{(2)} + S^{(4)} + \ldots$$

$$\mathcal{S}^{(2)} = \frac{\lambda}{2d^2\beta^3}\int dv \oint \frac{d\rho}{2\pi}\left[2\pi i\frac{d\xi}{d\rho}\right]\left[\frac{\partial X^i}{\partial \rho} + i\beta\rho^2\frac{\partial X^i}{\partial v}\right]\frac{\partial X^i}{\partial \xi},$$

$$S^{(4)} = \frac{\lambda}{2d^3\beta^5}\int dv \oint \frac{d\rho}{2\pi}\left[2\pi i\frac{d\xi}{d\rho}\right]^2\left[\frac{\partial X^i}{\partial \rho} + i\beta\rho^2\frac{\partial X^i}{\partial v}\right]\left[\frac{\partial X^i}{\partial \rho} + i\beta\rho^2\frac{\partial X^i}{\partial v}\right]\frac{\partial X^j}{\partial \xi}\frac{\partial X^j}{\partial \xi}$$

Boundary Conditions

- Solve resultant equations of motion (e.o.m) with imposing boundary conditions at $\rho = \rho_c$ to be

$$X^i(v, \rho_c + i\epsilon) = q^i_L(v) \ , \qquad X^i(v, \rho_c - i\epsilon) = q^i_R(v) \ . \quad (8)$$

- E.o.m are symmetric under $X^i \mapsto -X^i$, under which $q^i \mapsto -q^i$.
- Amplitude expansion of X^i has to be odd in q^i.
- The solution is $X^i = X^i_1 + X^i_3 + \dots$, where $X^i_{2k+1} \sim O(q^{2k+1})$.

$$\begin{aligned} &X^i_1(v, \rho_c + i\epsilon) = q^i_L(v), X^i_1(v, \rho_c - i\epsilon) = q^i_R(v), \\ &X^i_{2k+1}(v, \rho_c + i\epsilon) = 0, X^i_{2k+1}(v, \rho_c - i\epsilon) = 0. \end{aligned} \quad (9)$$

Solution & Boundary conditions

- The full linearised solution in frequency space becomes

$$\tilde{X}_1^i(\omega) = g_\omega[(1+f_\omega)q_R^i(\omega) - f_\omega q_L^i(\omega)] - g_{-\omega} e^{\beta\omega(1-\chi)} f_\omega [q_R^i(\omega) - q_L^i(\omega)]$$

  where $f_\omega = \frac{1}{e^{\beta\omega}-1}$ and $\chi = \frac{d}{2\pi i}\int_{\rho_c+i\varepsilon}^{\rho} dy\ \frac{y^{d-2}}{y^d-1}$.
- First term sources ingoing quasi-normal mode regular at future horizon, second term excites Hawking mode.
- In AdS_3 (d=2), the bulk to boundary Green's function is

$$g_\omega = \left(\frac{\rho_c}{\rho}\right)\frac{\omega + i r_h \rho}{\omega + i r_h \rho_c}\,.$$

Table of Contents

Quadratic effective action

- The full quadratic action in derivative expansion,

$$
\begin{aligned}
S^{(2)} + S^{(2)}_{ct} =& m_p \int \frac{d\omega}{2\pi}(\omega^2 + i\gamma\omega) q^i_d(-\omega) q^i_a(\omega) \\
&+ \frac{i}{2} m_p^2 \int \frac{d\omega}{2\pi} \left(\langle f^2 \rangle - \omega^2 Z_I\right) q^i_d(-\omega) q^i_d(\omega) \,.
\end{aligned}
$$

- The counter-term is given by,

$$
S^{(2)}_{ct} = \left(m_p(T=0) - \frac{r_h \rho_c}{2\pi\alpha'} \right) \int \frac{d\omega}{2\pi} \omega^2 q^i_a(\omega) q^i_d(-\omega) \,.
$$

Parameters in quadratic effective action

- [Thermal mass] : $m_p = m_p(T=0) - \frac{\lambda}{8\pi d\beta}\left(1 + \int_1^\infty \frac{t^{d-4}-1}{t^d-1} dt\right)$
- [colour strength] :
  $m_p^2 Z_I = -\frac{\lambda}{2d^2\beta}\left[\frac{1}{6} + \left(\frac{d}{2\pi}^2\right)\int_1^\infty \frac{t^{d-4}dt}{t^d-1}\int_1^t \frac{y^{d-4}dy}{y^d-1}(y^2-1)\right]$
- [damping factor] : $m_p\gamma = \frac{\lambda}{2d^2\beta^2}$
- [strength of the quadratic noise] : $m_p^2\langle f^2\rangle = \frac{\lambda}{d^2\beta^3}$
- Consistent with FDR: $\gamma = \frac{1}{2}\beta m_p\langle f^2\rangle$.

Dimension dependence of mass and the color strength

d	$m_p(T)$ in units of $\frac{\lambda}{8\pi d\beta}$	Z_I in units of $\frac{\lambda}{2d^2\beta}$
2	0	-0.217
3	$\frac{1}{18}(9\ln 3 - \sqrt{3}\pi) - 1 \approx -0.753$	-0.234
4	-1	-0.256
5	$\approx$ -1.115	-0.282
6	$\frac{1}{12}(3\ln 3 - \sqrt{3}\pi) - 1 \approx -1.179$	-0.314

Nonlinear FDR

- The quartic effective couplings are

$$m_p^3\zeta_\gamma = (7-d)\frac{\lambda}{d^3\beta^4}\,, \qquad m_p^4\zeta_N = -12(7-d)\frac{\lambda}{d^3\beta^5}$$

- The generalised fluctuation-dissipation-relation is the following

$$\zeta_\gamma = -\frac{1}{12}\beta m_p \zeta_N \ . \tag{10}$$

Table of Contents

Summary of the talk

- We have studied the leading non-linear corrections to the Brownian motion of a heavy quark probing a strongly coupled CFT plasma using holographic SK path integral.
- The influence phase of the quark gives rise to a non-linear Langevin equation with a non-Gaussian noise.
- The local effective theory obeys recently developed non-linear fluctuation dissipation relation that relates the non-Gaussianity of thermal noise to the thermal jitter in the damping constant of the Brownian particle.

Future works

- It would be interesting to extend the analysis presented in this paper to account for arbitrary initial states.
- A possible extension would be to study the effects of backreaction of the string on the black brane geometry. This corresponds in the CFT to the energy disturbances in the plasma created by the moving quark.
- It will be useful to study field theories in the black brane backgrounds and derive dual open quantum field theories, by integrating out the effect of the black brane.

Thank you.