Quantum Gravity & Quantum Information

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Quantum Gravity & Quantum Information

$\boxed{1}$	Two Puzzles in Black Hole Information	QG
		$\downarrow$
$\boxed{2}$	Page Curve and Recent Results	QI
		$\downarrow$
$\boxed{3}$	Typical Black Hole Entanglement Entropy	QG

$\boxed{1}$ Two Puzzles in Black Hole Information

i) What is the origin of black hole entropy?

$$S_{BH} = \frac{A_{BH}}{4G\hbar}$$

ii) What is the fate of black hole information?

$$A_{BH}(t) \longrightarrow 0 \qquad \text{evaporation}$$

$$\begin{cases} S_{BH}(t) \longrightarrow 0 \\ S_{rad}(t) \longrightarrow \infty \end{cases}$$

(i) Black Hole Entropy (Bekenstein, Hawking '74)

Assumptions:
- Equilibrium
- Vacuum & Isolated
- here non-rotating for simplicity

- Thermodynamic derivation

a) GR — Horizon Area — $A_{BH} = 4\pi R^2$ with $R = 2GM/c^2$

b) QFT — Temperature — $k_B T = \dfrac{\hbar c}{4\pi R}$

c) Thermodyn. — Entropy — $\delta S = \dfrac{\delta Q}{T}$

$\Downarrow$

Entropy
$$\boxed{\delta S_{BH} \underset{(c)}{=} \frac{\delta(Mc^2)}{T} \underset{(a),(b)}{=} k_B \frac{\delta A_{BH}}{4G\hbar/c^3}}$$

* Puzzle:

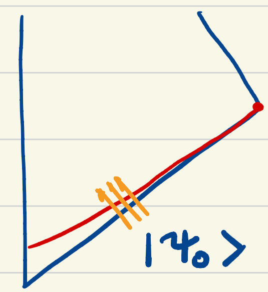
- For Non-Grav Systems: Thermodynamics $\longleftarrow$ Stat. Mech $\longleftarrow$ Pure State QM: Entropy from Entanglement
- For BH: S_{BH} $\longleftarrow$? $\longleftarrow$?

ii) Fate of Black Hole Information

(Hawking '76)
(Page '93)

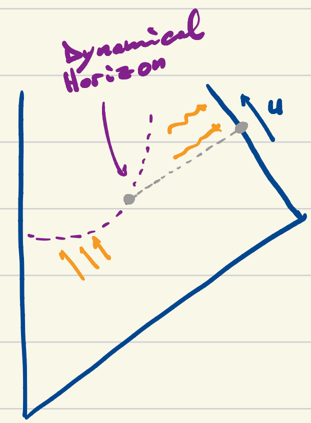
- Initial Conditions:

  pure state of grav. & matter



- Isolated, unitary evolution

  BH formation $\quad S_{BH}$



- Evaporation

$$\begin{cases} S_{rad}(u) & \text{increases} \\ S_{BH}(u) & \text{decreases} \\ & \\ S_{tot}(u) = S_{BH}(u) + S_{rad}(u) & \text{increases} \end{cases} \qquad \text{(GSL)}$$

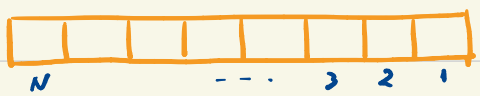
- Hawking Information Puzzle: $\quad M_B(u) \to 0\,, \quad |\Psi_0\rangle \longrightarrow \rho_{rad} \quad ?$

- Page Information Puzzle: $\quad M_B(u) < \frac{M_B(u_0)}{\sqrt{2}}\,, \quad$ profile of $S_{rad}(u) \quad ?$

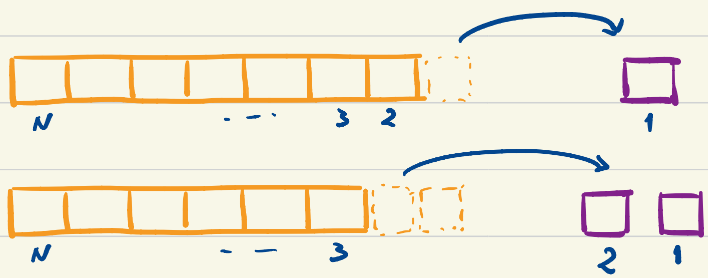
Page Curve: Qubit Model of Evaporation (Page '93)

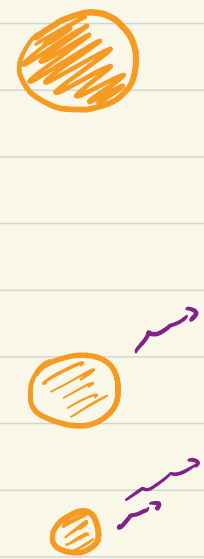
- N qubits in a pure state

$$|\psi\rangle = \sum_{i_1 \ldots i_N = \pm 1} c_{i_1 \ldots i_N} |i_1\rangle \cdots |i_N\rangle$$



- Extract one qubit at a tim





BH formation

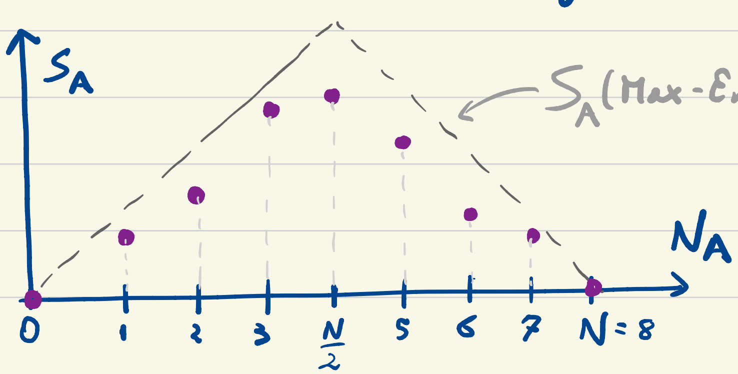
BH evaporation

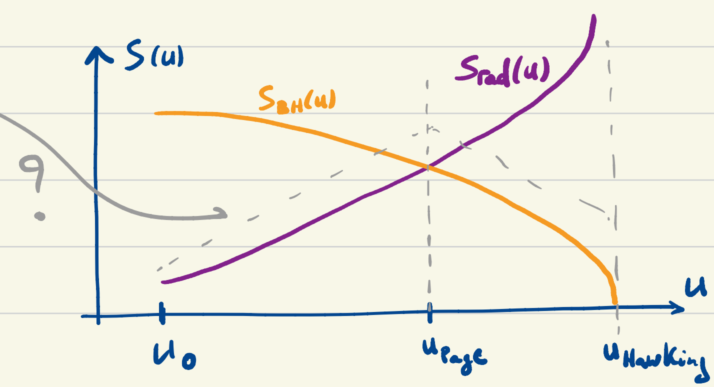
- Entanglement Entropy

$$S_A(|\psi\rangle) = -\mathrm{Tr}_A(\rho_A \log \rho_A) \quad \text{with} \quad \rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

$\shortparallel$

S_B





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Quantum Gravity & Quantum Information

$\boxed{1}$ Two Puzzles in Black Hole Information — QG

$\boxed{2}$ $\boxed{\text{Page Curve and Recent Results}}$ — QI

$\boxed{3}$ Typical Black Hole Entanglement Entropy — QG

(i) Typical Entropy of Random States

(ii) Time Evolution and Thermalization

(iii) Typical Entropy with Constraints

(iv) Typical Entropy of Sub-Algebras

$\boxed{2}$ (i) Page Curve and Typical Entropy of Random States

- Hilbert Space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$d_A = \dim \mathcal{H}_A$
$d_B = \dim \mathcal{H}_B$

- Random Pure State $|\psi\rangle = U |\psi_0\rangle$

Reference state

Random Unitary
from uniform probability distribution $d\mu(U)$ (Haar Measure)

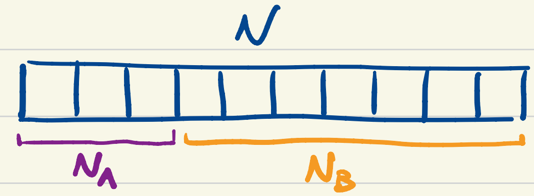
- Entanglement Entropy $S_A(|\psi\rangle)$

* What is the probability of finding S_A ?
→ compute $P(S_A)\, dS_A$

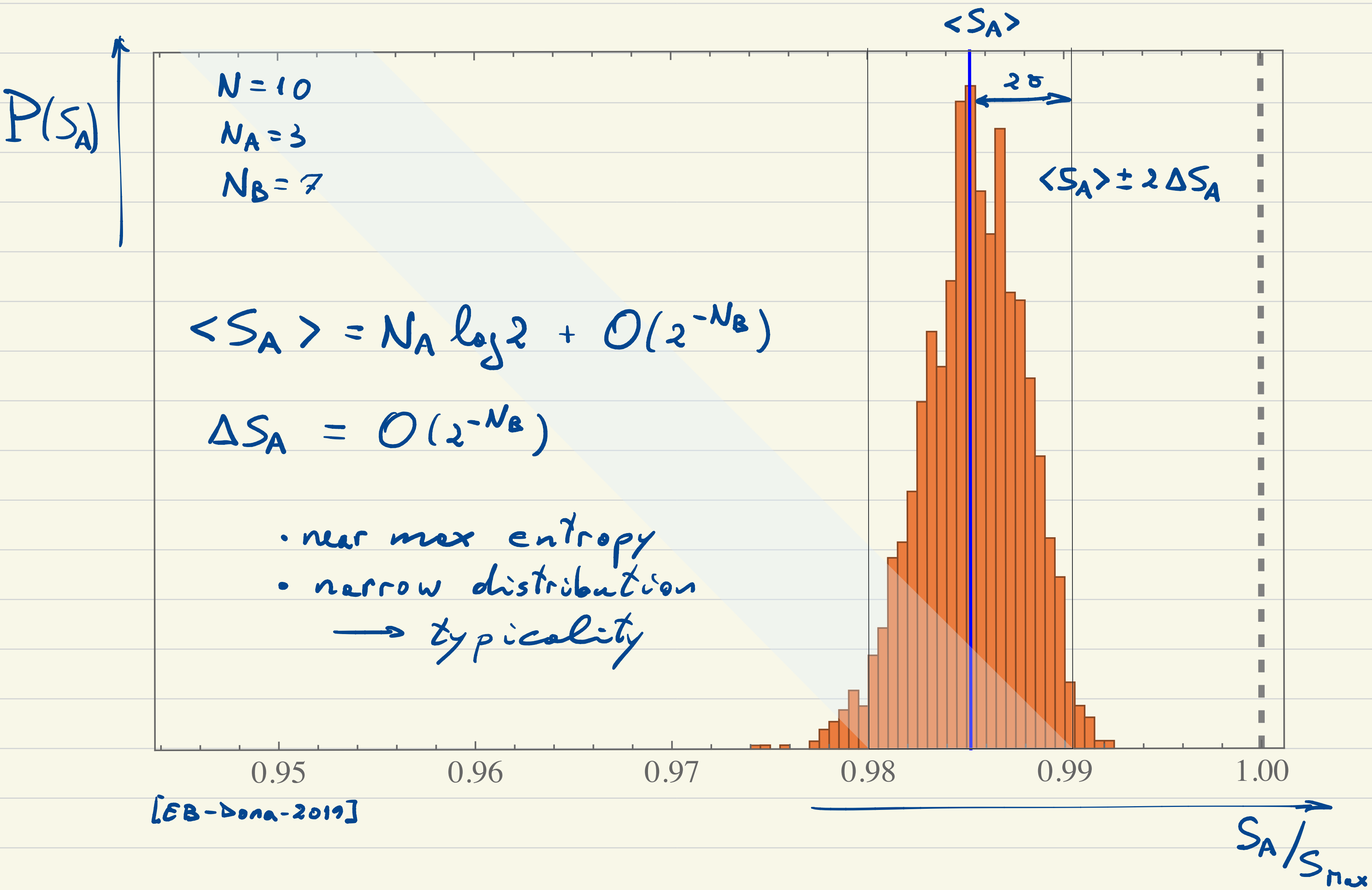
- Page '93 : Average Entropy of a Subsystem $\langle S_A \rangle = \int S_A(U|\psi_0\rangle)\, d\mu(U)$
- Bianchi-Donà '19 : Typical entropy $\langle S_A \rangle \pm \Delta S_A$

from moments $\mu_n = \int (S_A)^n \, P(S_A)\, dS_A$

Example: N qubits



$d_A = 2^{N_A}$, $d_B = 2^{N_B}$



$$\langle S_A \rangle = N_A \log 2 + O(2^{-N_B})$$

$$\Delta S_A = O(2^{-N_B})$$

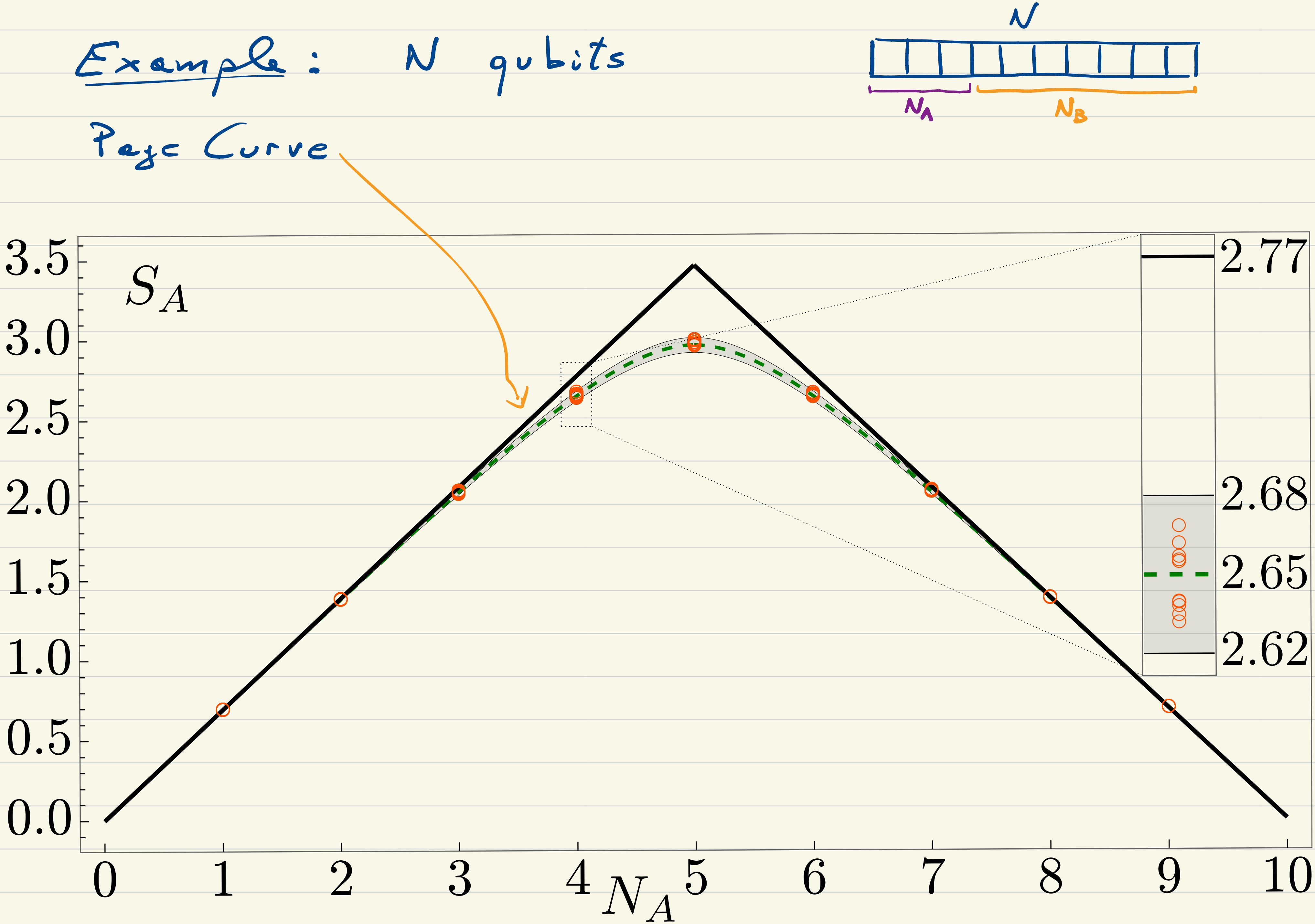
- near max entropy
- narrow distribution

$\longrightarrow$ typicality

[EB-Dona-2019]

Example: N qubits

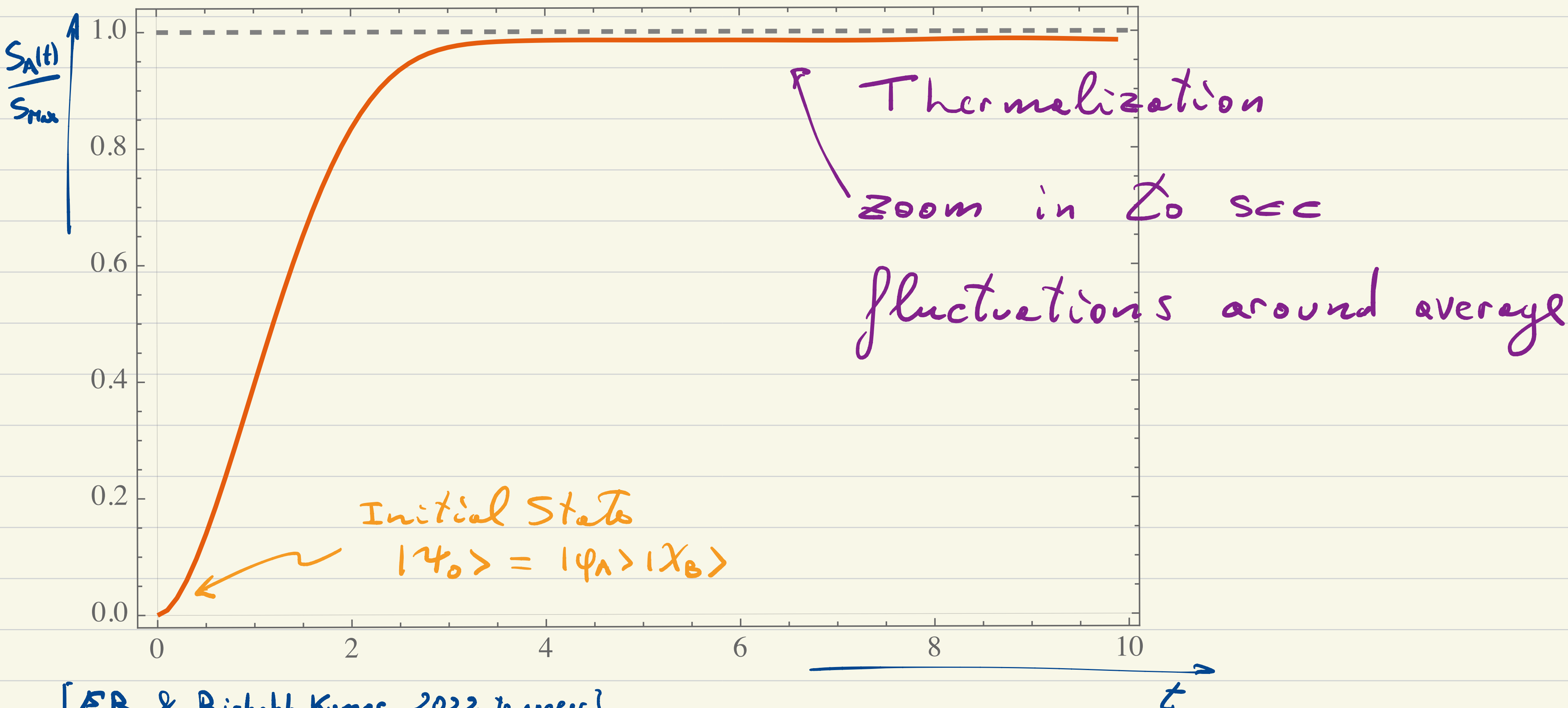
Page Curve



[EB-Dona-2019]

(ii) Time Evolution of the Entanglement Entropy

- Initial Factorized State $|\psi_0\rangle = |\varphi_A\rangle|\chi_B\rangle$
- Evolution with Random Hamiltonian $H = \sum_{ij} H_{ij} |i\rangle\langle j|$
- Entanglement Entropy $S_A(t)$ of $|\psi_t\rangle = e^{iHt}|\psi_0\rangle$



[EB & Rishabh Kumar, 2022 to appear]

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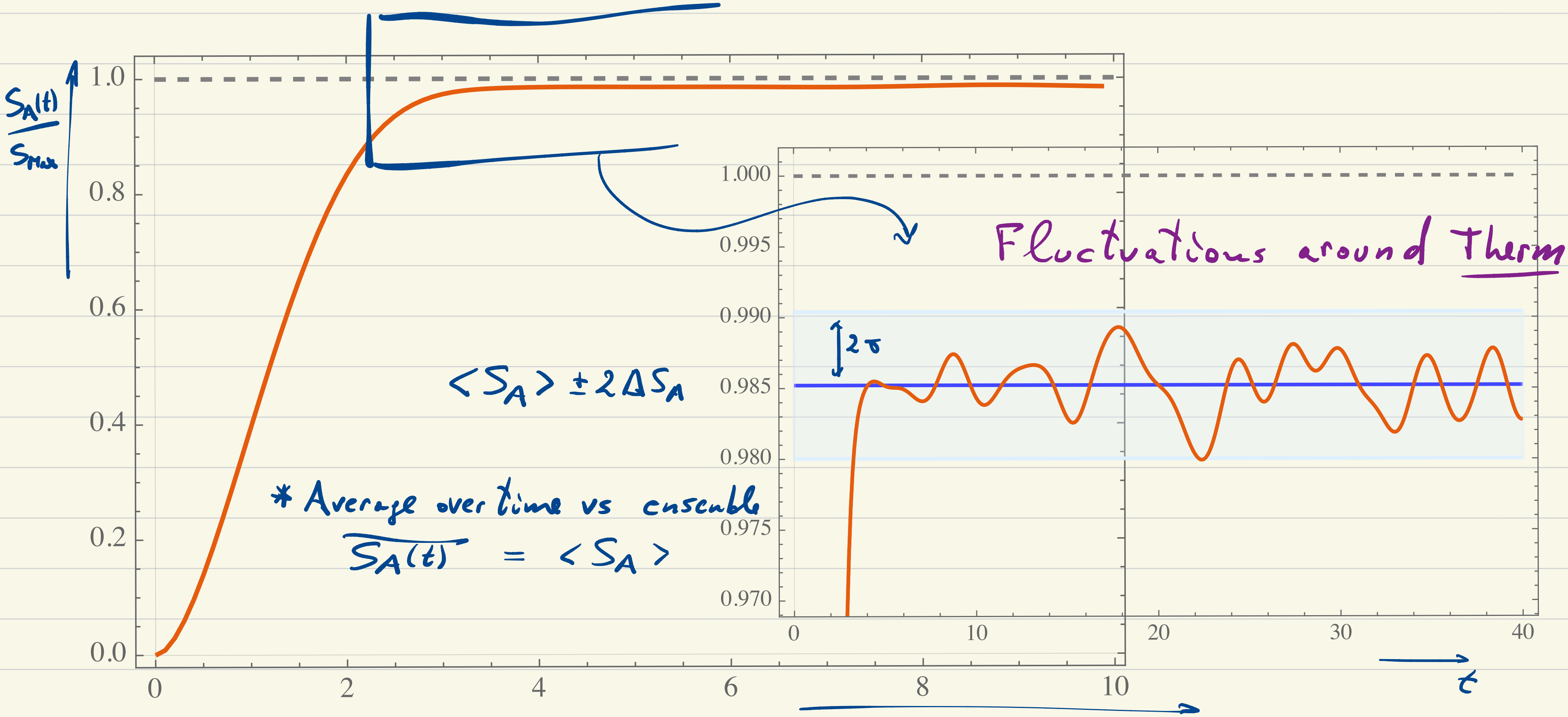


[EB & Rishabh Kumar, 2022 to appear]

(iii) Typical Entanglement Entropy of Random States with Constraints

- Hilbert Space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Constraint $|\psi_E\rangle \in \mathcal{H}_E \subset \mathcal{H}$

  e.g. • Hamiltonian $H = H_A + H_B$

  • Energy Eigen-space $H|\psi_E\rangle = E|\psi\rangle$

  • Decomposition as direct sum

$$\boxed{\mathcal{H}_E = \bigoplus_{\varepsilon} \left(\mathcal{H}_{A,\varepsilon} \otimes \mathcal{H}_{B,E-\varepsilon} \right)}$$

- Entanglement Entropy $S_A(|\psi_E\rangle)$
- Probability distribution $\boxed{P_E(S_A)\, dS_A}$

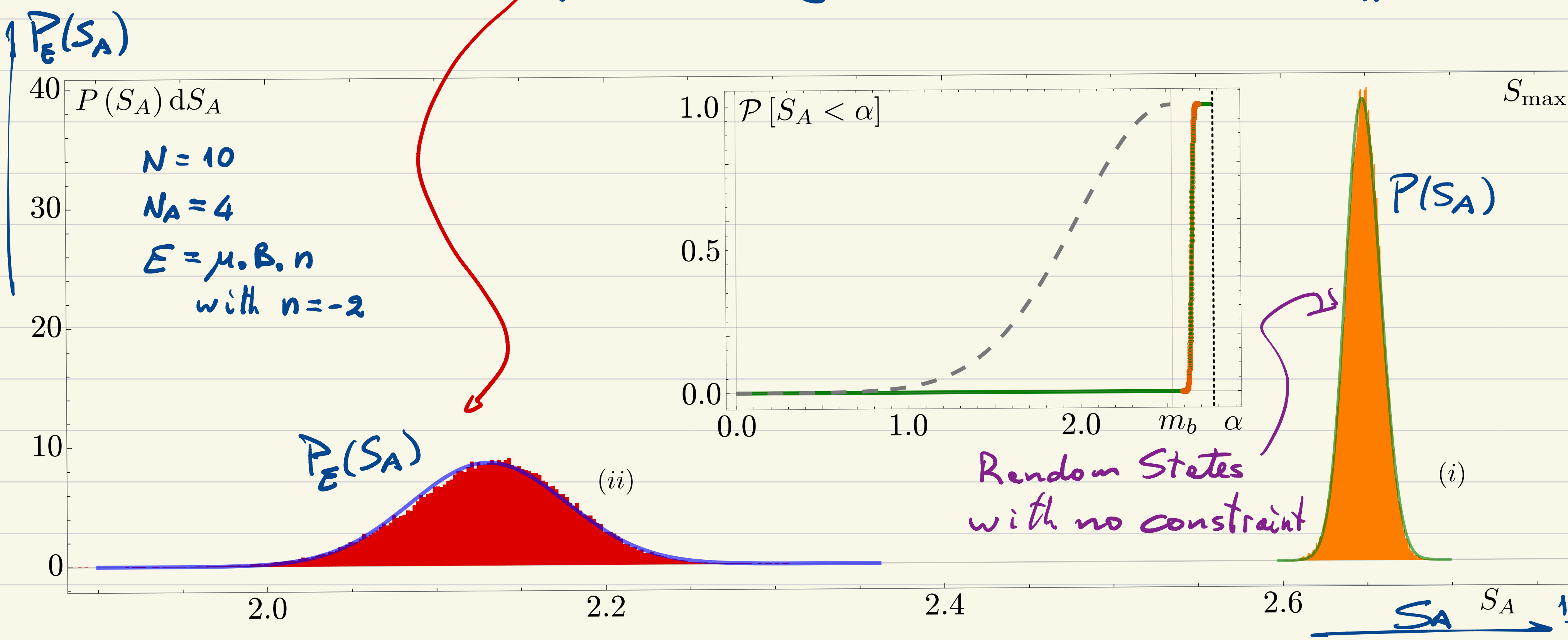
EB-Donà, 2019

EB-Hackl-Kieburg, 2021

EB-Hackl-Kieburg-Rigol-Vidmar, 2021

Example: typical entanglement in a paramagnet

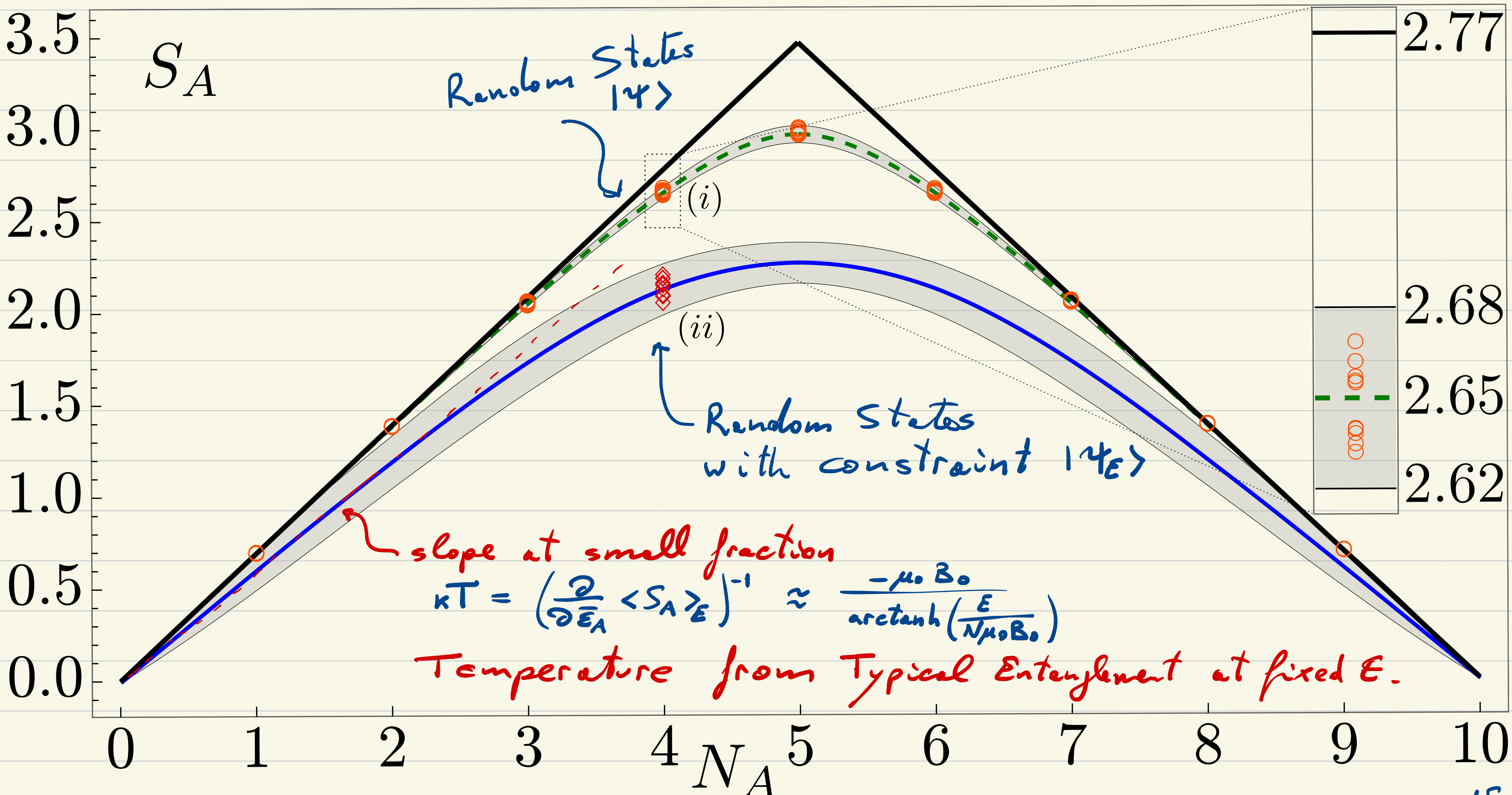
- N spins in a uniform magnetic field $\vec{B}$
- Hamiltonian $\quad H = \sum_{i=1}^{N} \mu_0 \vec{B} \cdot \vec{S}_i$
- Random Eigenstate $|\psi_E\rangle$ of energy E
- Probability of finding entanglement entropy S_A



[EB-Donà, 2019]

Example: typical entanglement in a paramagnet

- Random Eigenstate $|\psi_E\rangle$ of energy E
- Page Curve



(iv) Entanglement Entropy of Sub-Algebras of Observables

- Generally, $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_B$, the Hilbert space does not come with a decomposition in subsystems

- Physically, measurements probe only part of the system

  $\to$ sub-algebra of observables $\mathcal{O}_A \in \mathcal{A}_A \subset \mathcal{A}$

  $\to$ not measured: commutant $\mathcal{A}_B = \{\mathcal{O}_B \in \mathcal{A} \mid [\mathcal{O}_B, \mathcal{O}_A] = 0 \ \forall \mathcal{O}_A\}$

  $\to$ center $\mathcal{Z} = \mathcal{A}_A \cap \mathcal{A}_B$

- Decomposition of the Hilbert Space

eigenspaces of $\mathcal{Z}$

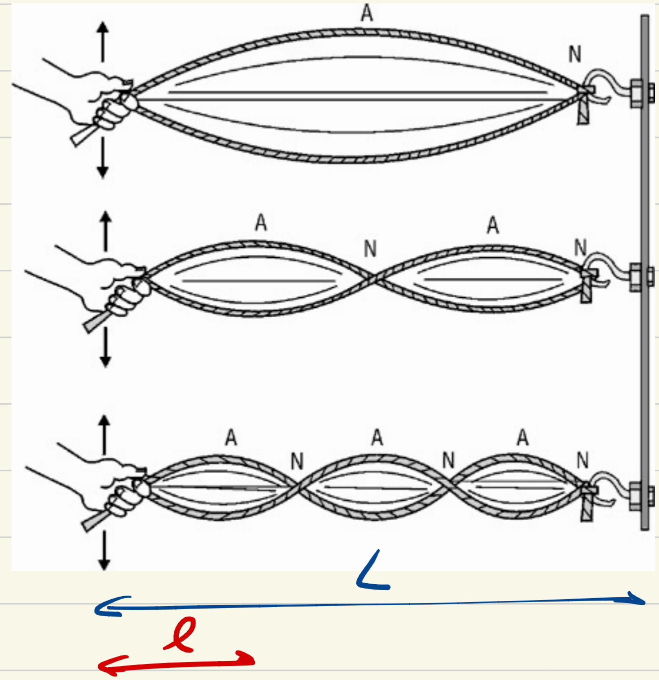
$$\mathcal{H} = \bigoplus_{\varsigma} \left(\mathcal{H}_A(\varsigma) \otimes \mathcal{H}_B(\varsigma) \right)$$

- Typical Entropy of SubAlgebra $P(S[\mathcal{O}_A])$

$$\boxed{\langle S[\mathcal{O}_A] \rangle = \sum_{\varsigma} \frac{d_\varsigma}{d} \left(\langle S_\varsigma \rangle + \Psi(d+1) - \Psi(d_\varsigma + 1) \right)}$$

[EB-Satz, 2019]
[EB-Donà, 2019]

▪ Example: Electromagnetic Field & Black Body Radiation



- EM field in a box of volume L^3
- Pure state $|\psi_E\rangle$ of energy E
- Measurement: Antenna of length ℓ

  ↳ subalgebra $\mathcal{A}_A$

- Typical Entanglement Entropy

$$\langle S[\mathcal{O}_A]\rangle_E \simeq \frac{4\sqrt{\pi}}{3(15)^{1/4}}\left(\frac{E_A\,\ell}{\hbar c}\right)^{3/4}$$

↳ Black Body Entropy

- Temperature $kT = \left(\frac{\partial}{\partial E_A}\langle S\rangle_E\right)^{-1}$

[EB - Donà - Muino, 2022 to appear]

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Typical Black Hole Entanglement Entropy

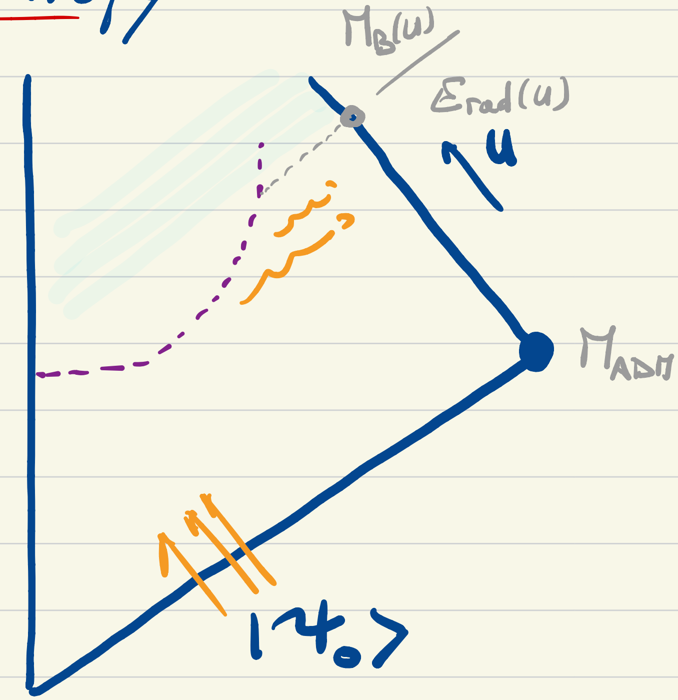
- Evaporating Black Hole

$$M_{ADM} = M_B(u) + E_{rad}(u)$$

- Assume pure state for Grav + Matt

$$|\psi_{M_{ADM}}\rangle \quad \text{at fixed } M_{ADM}$$

- Algebra of Observables $g_{\mu\nu}$, φ



- Sub-algebra: $g_{\mu\nu}$ only. Center $\mathcal{Z}$: $\left\{ \begin{array}{l} \hat{M}_B(u) \\ \hat{E}_{rad}(u) - M_{ADM} \end{array} \right.$

Average Entanglement Entropy of Sub-Algebra

- use $\dim \mathcal{H}_{BH}(M_B) \approx \int Z(\beta)\, e^{+\beta M_B} \frac{d\beta}{2\pi i} \simeq e^{+4\pi \frac{G M_B^2}{\hbar c}}$

Gibbons-Hawking '77
Brown-York '93
Sen '12
EB-Haggard '18

- $\langle S[\mathcal{O}_A] \rangle$ formula in terms of dim (EB-Dona '19)

$$\Rightarrow \quad \boxed{\langle S_{BH} \rangle \approx 4\pi \frac{G M_B(u)}{\hbar c} = \frac{A_{BH}(u)}{4 G \hbar / c^3}}$$

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