

Foundations of Gravitational waves in cosmology

Béatrice Bonga – 4 Feb 2022 – Second Chennai Symposium
on Gravitation and Cosmology
[BB+Prabhu, arXiv:2009.01243]

Radboud University



Invaluable tool: perturbation theory

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$$



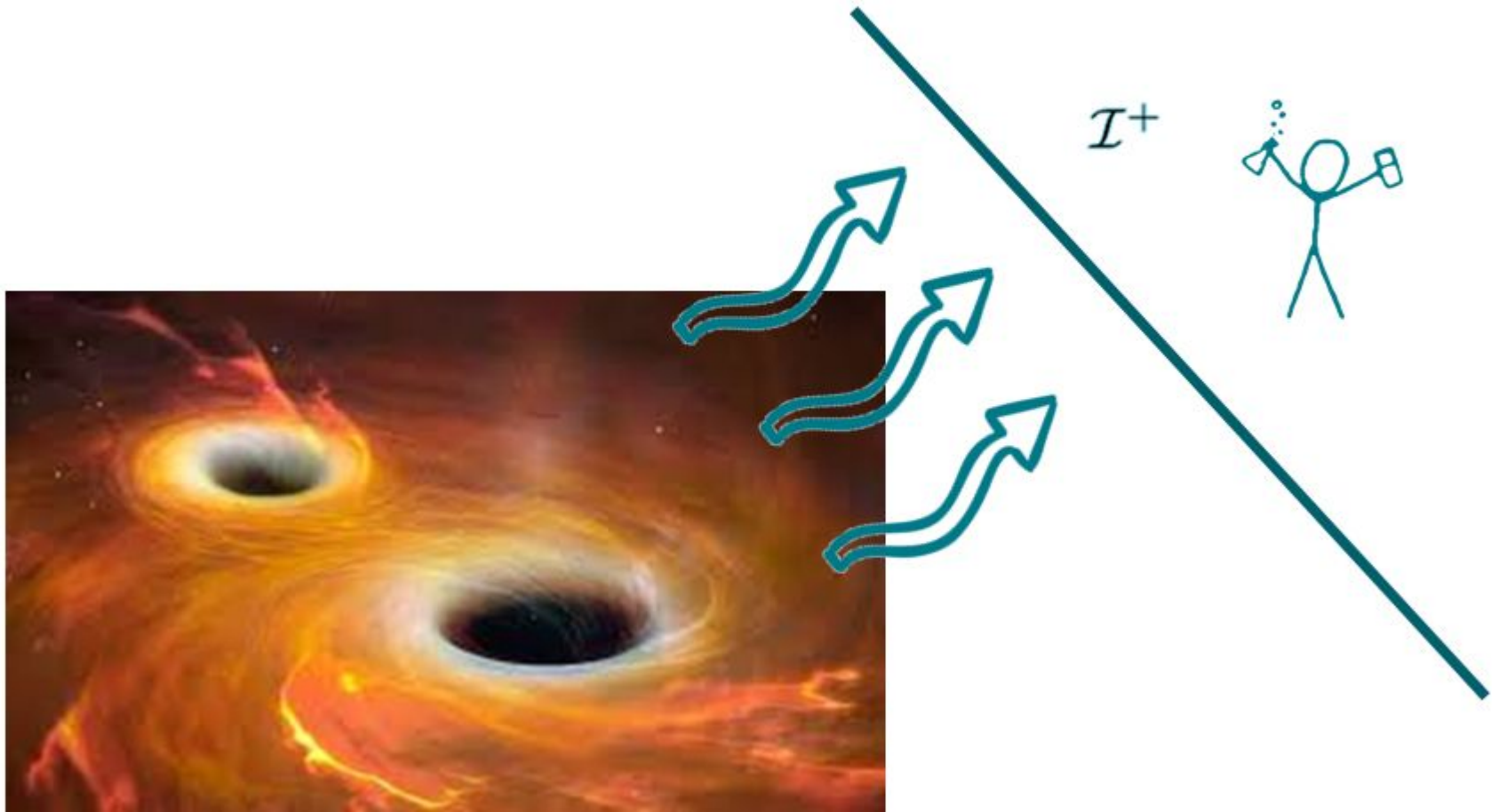
Minkowski spacetime
Kerr spacetime
FLRW spacetime

⋮

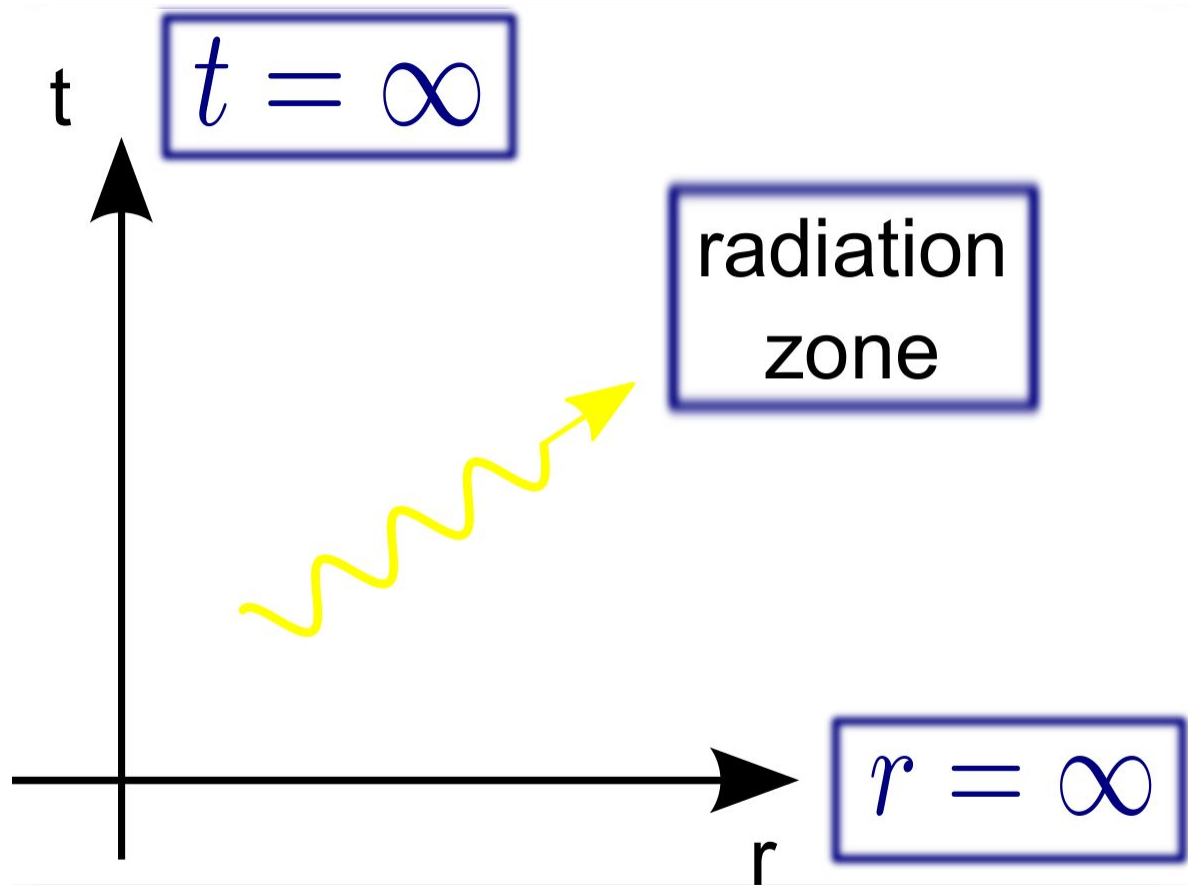
But there is no canonical split!

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$$

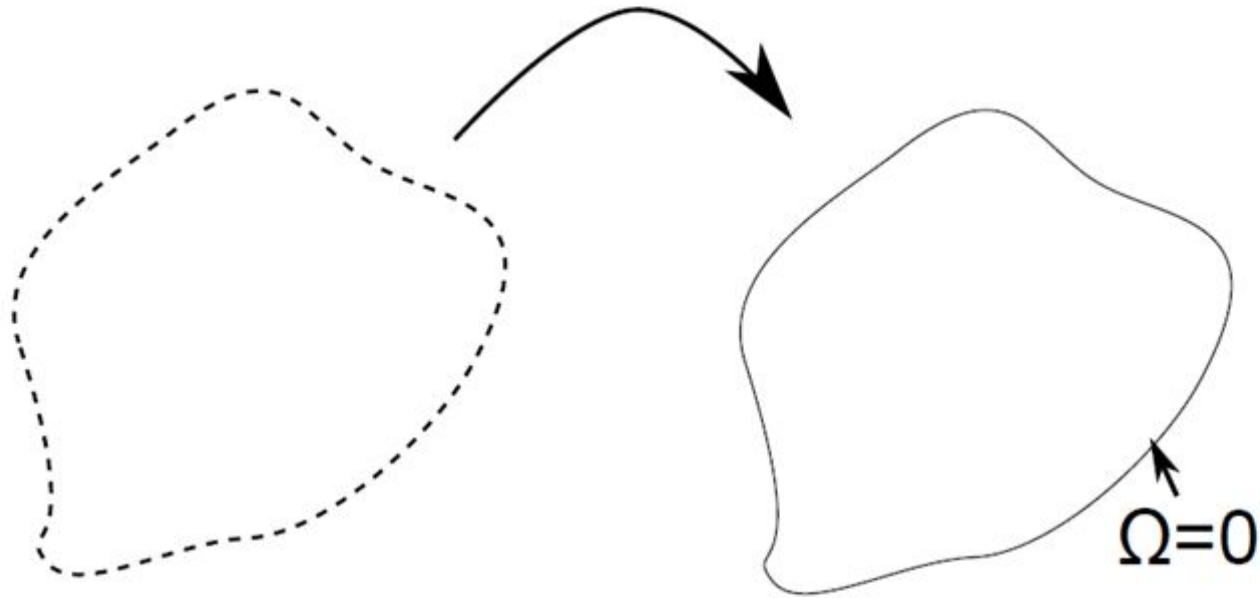
From messy physics to peaceful realm



Different infinities



Key idea: bring infinity to a finite distance

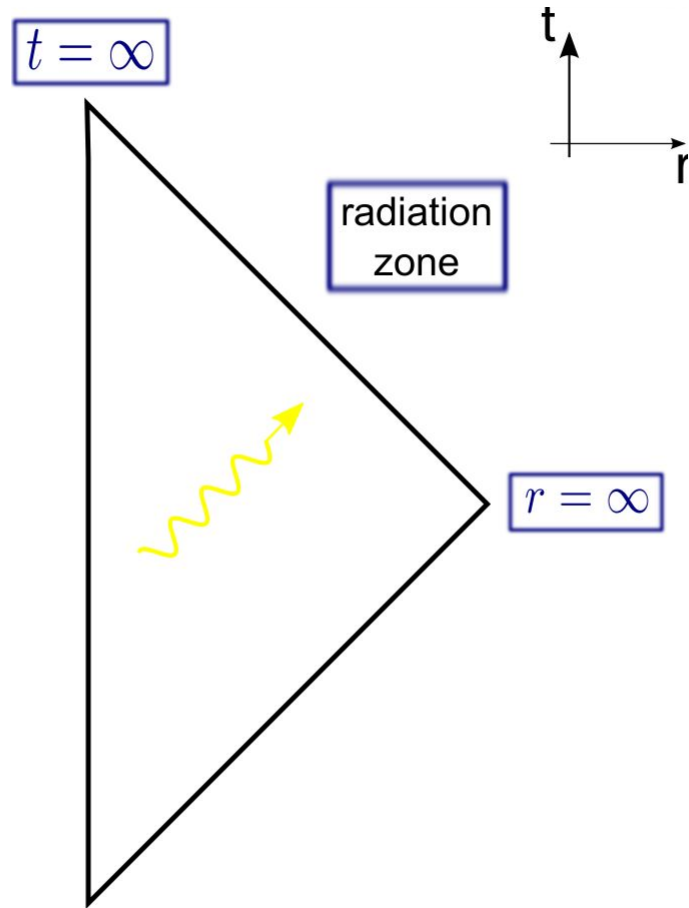


$$(\hat{M}, \hat{g}_{ab})$$

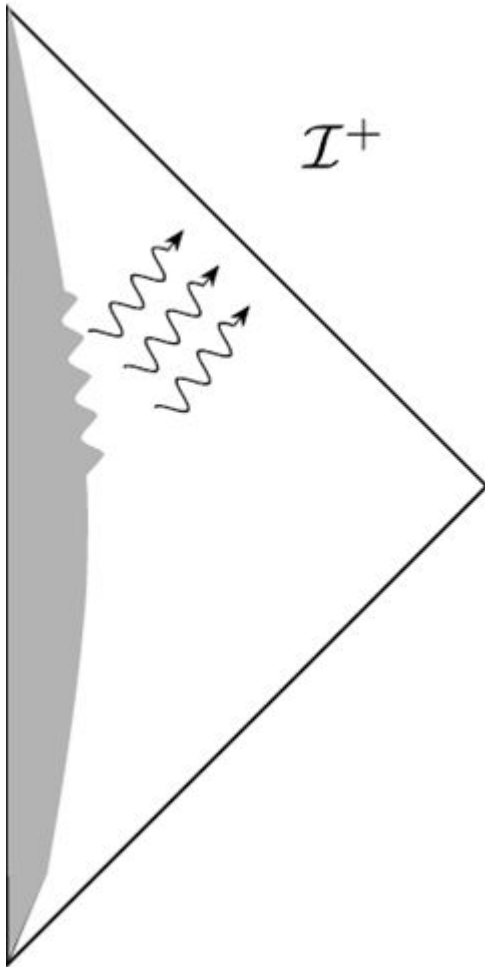
$$(M, g_{ab} = \Omega^2 \hat{g}_{ab})$$

Conformal completion

Conformal diagram Minkowski



Asymptotic flatness

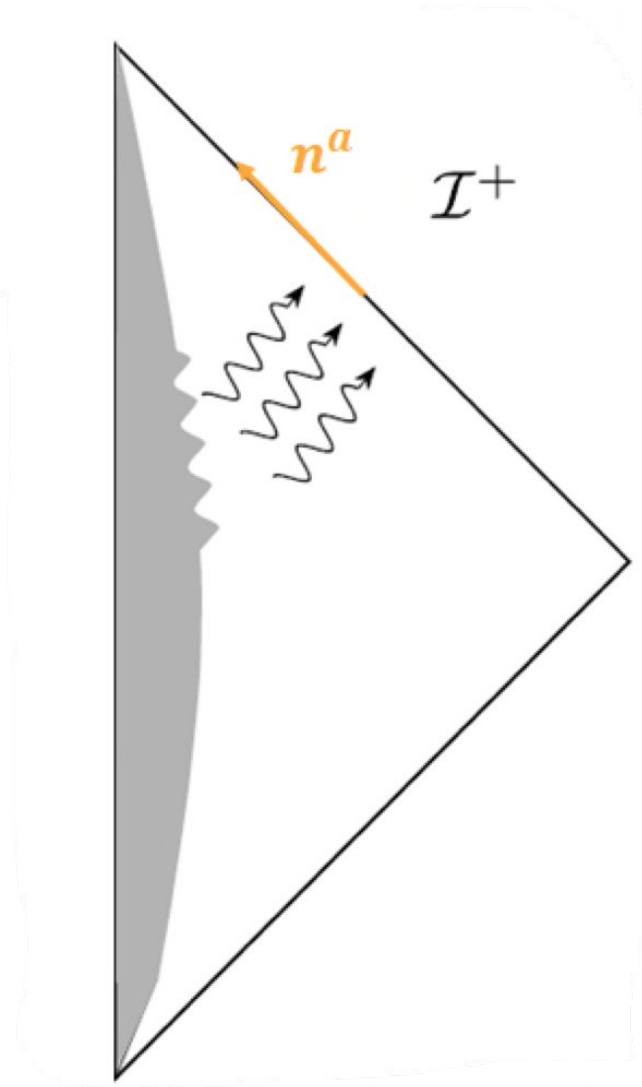


A physical spacetime (\hat{M}, \hat{g}_{ab}) is asymptotically flat if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{J} \cong \mathbb{R} \times \mathbb{S}^2$ such that

1. Ω and $g_{ab} = \Omega^2 \hat{g}_{ab}$ are smooth on M , $\Omega \hat{=} 0$ and $n_a = \nabla_a \Omega$ is nowhere vanishing on \mathcal{J}
2. Einstein's equations are satisfied with \hat{T}_{ab} such that $\Omega^{-2} \hat{T}_{ab}$ has a smooth limit to \mathcal{J}

Consequences

- Einstein's equation $\implies n^a$ is null on \mathcal{I}
- q_{ab} = induced metric on \mathcal{I} is degenerate: 0 + +



Universal structure

This is common to all asymptotically flat spacetimes

$$\{g_{ab}, n^a\} = \{\omega^2 g_{ab}, \omega^{-1} n^a\}$$

Gravitational radiation is encoded in the next-order structure and differs from spacetime to spacetime

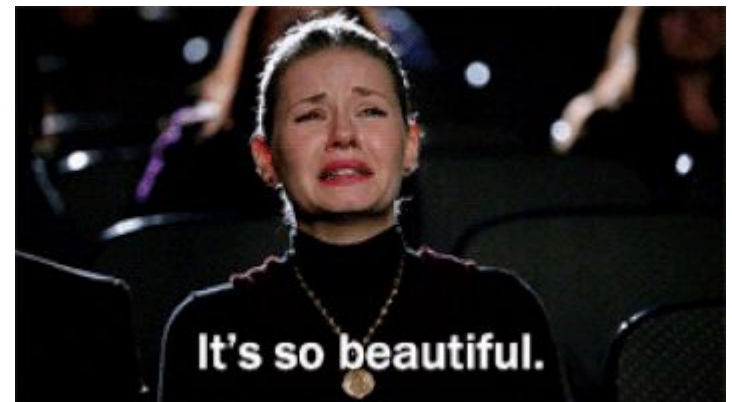
Key points of asymptotics

Nowhere in this construction did we introduce a split of the background and “gravitational waves”.

The split occurs naturally at null infinity:

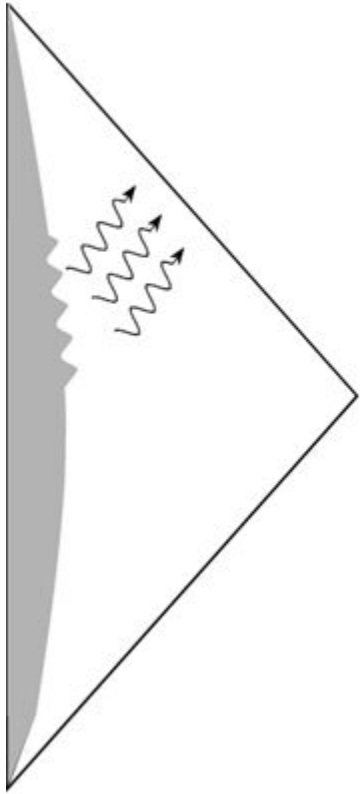
- universal structure is like a background,
- first order structure contains gravitational radiation,

and it is fully non-linear!



Generic asymptotically flat spacetime

$$d\hat{s}^2 = -UV du^2 - 2U dudr + \gamma_{AB}(r d\theta^A + W^A du)(r d\theta^B + W^B du)$$



$$U = 1 + B/r^2 + O(r^{-3}),$$

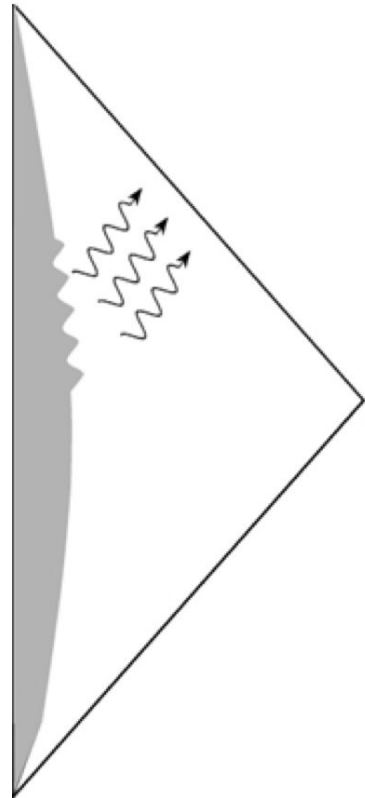
$$V = 1 - 2M/r + N/r^2 + O(r^{-3}),$$

$$W^A = A^A/r + B^A/r^2 + O(r^{-3}),$$

$$\gamma_{AB} = \Omega_{AB} + f_{AB}/r + \frac{1}{4}f^2\Omega_{AB}/r^2 + O(r^{-3})$$

Generic asymptotically flat spacetime

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Flat Space

Mass aspect

Radiative modes

Angular momentum aspect

Two definitions are equivalent!

**Geometric description
à la Penrose
(with the conformal completion)**

**Coordinate description
à la Bondi & Sachs**

Asymptotic symmetry algebra

Spacetime diffeomorphism that leave the universal structure at null infinity invariant

Universal structure

$$\mathcal{L}_\xi q_{ab} \hat{=} 2 \alpha q_{ab} \text{ with } \mathcal{L}_n \alpha \hat{=} 0$$

$$\mathcal{L}_\xi n^a \hat{=} -\alpha n^a$$

Coordinates

$$\Omega^2 \mathcal{L}_\xi \hat{g}_{ab} \hat{=} 0$$



Bondi-Metzner-Sachs algebra (BMS)

- Asymptotic symmetry algebra is **bigger** than Poincaré
- BMS = supertranslations & rotations

$$\xi^a \partial_a = \left(f(\theta, \varphi) + \frac{1}{2} u D_A Y^A \right) \partial_u + Y^A \partial_A$$

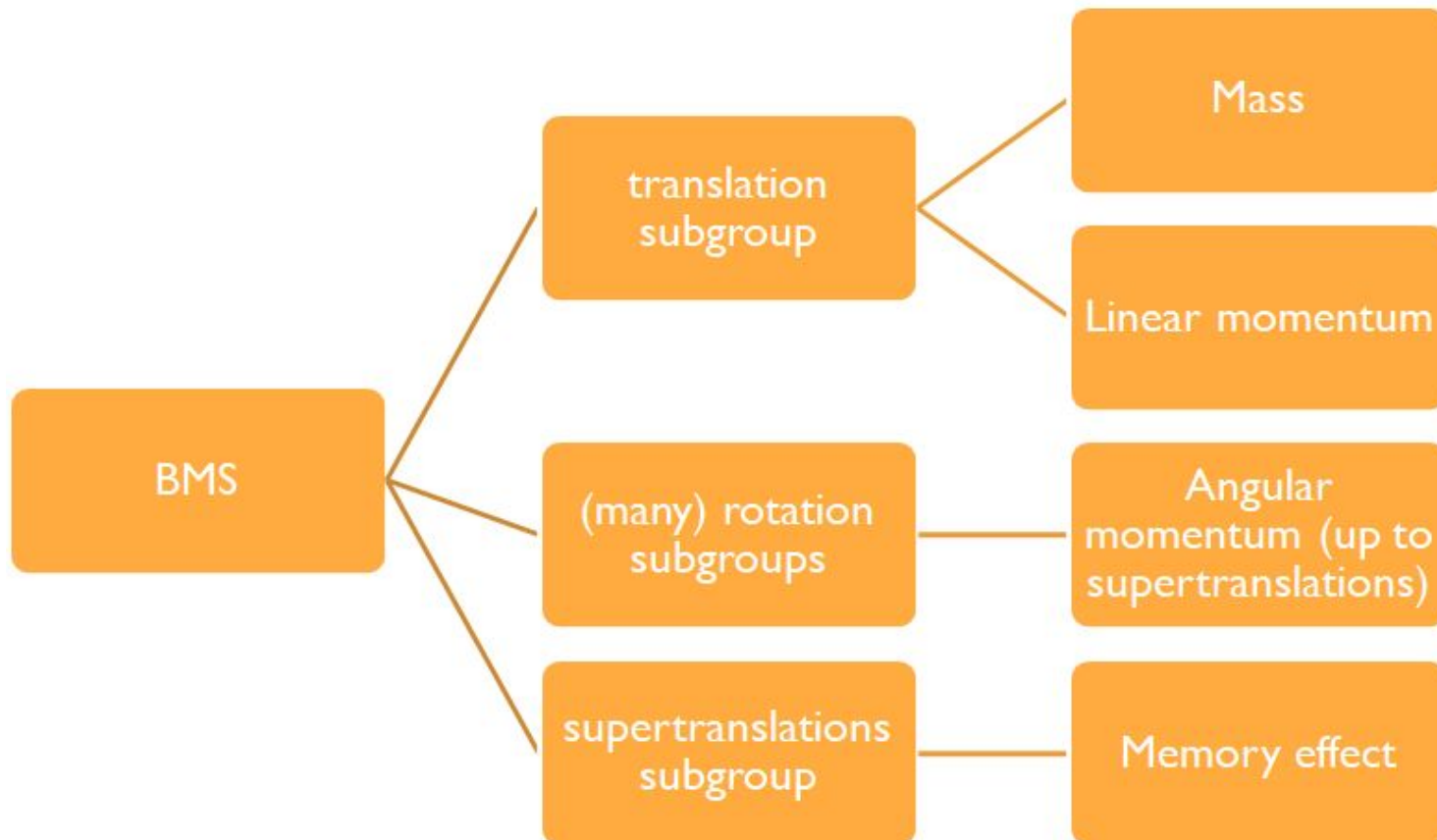
supertranslations

rotations

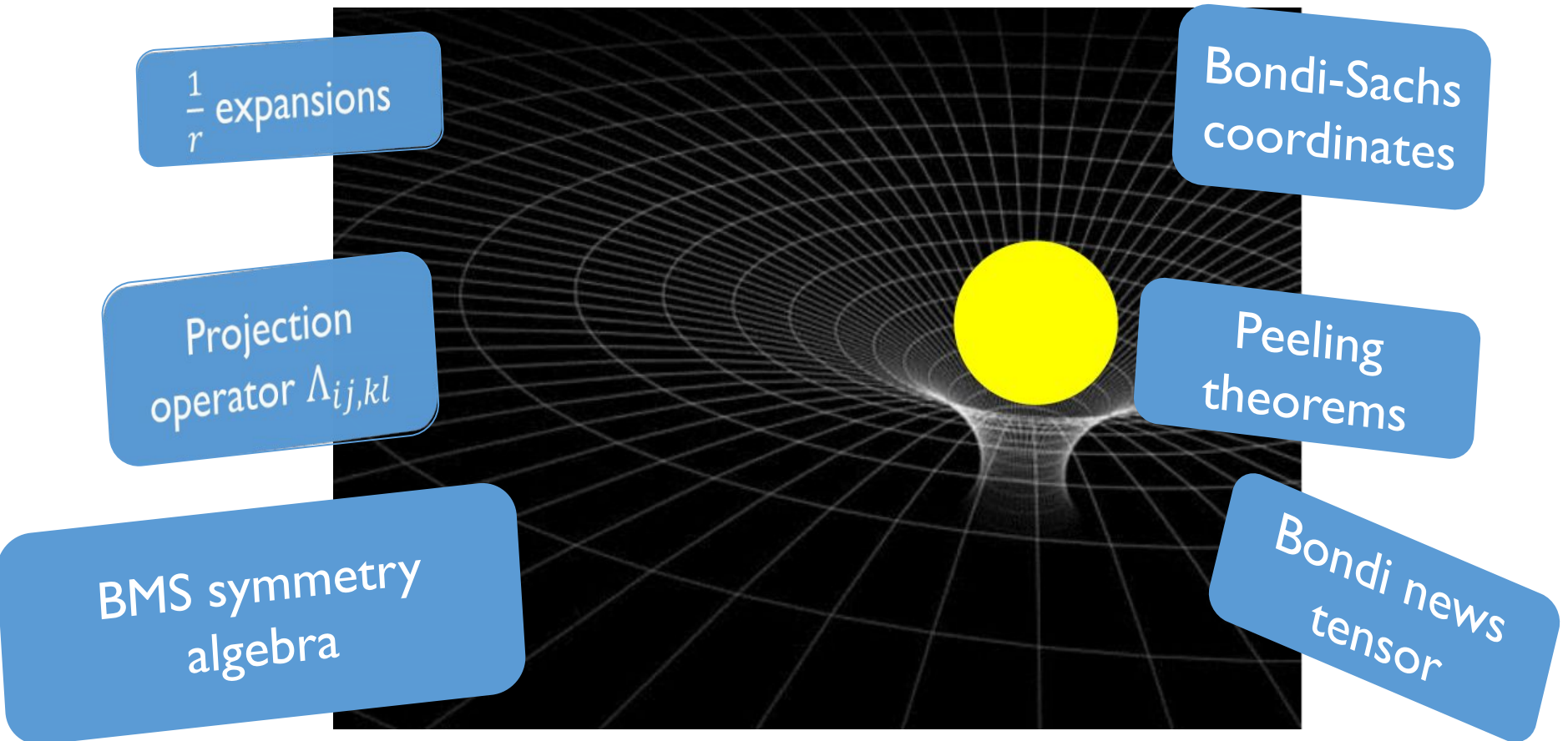
$$2D_{(A} Y_{B)} + q_{AB} D_C Y^C = 0$$

What is BMS good for?

It provides quantities with a physical interpretation!

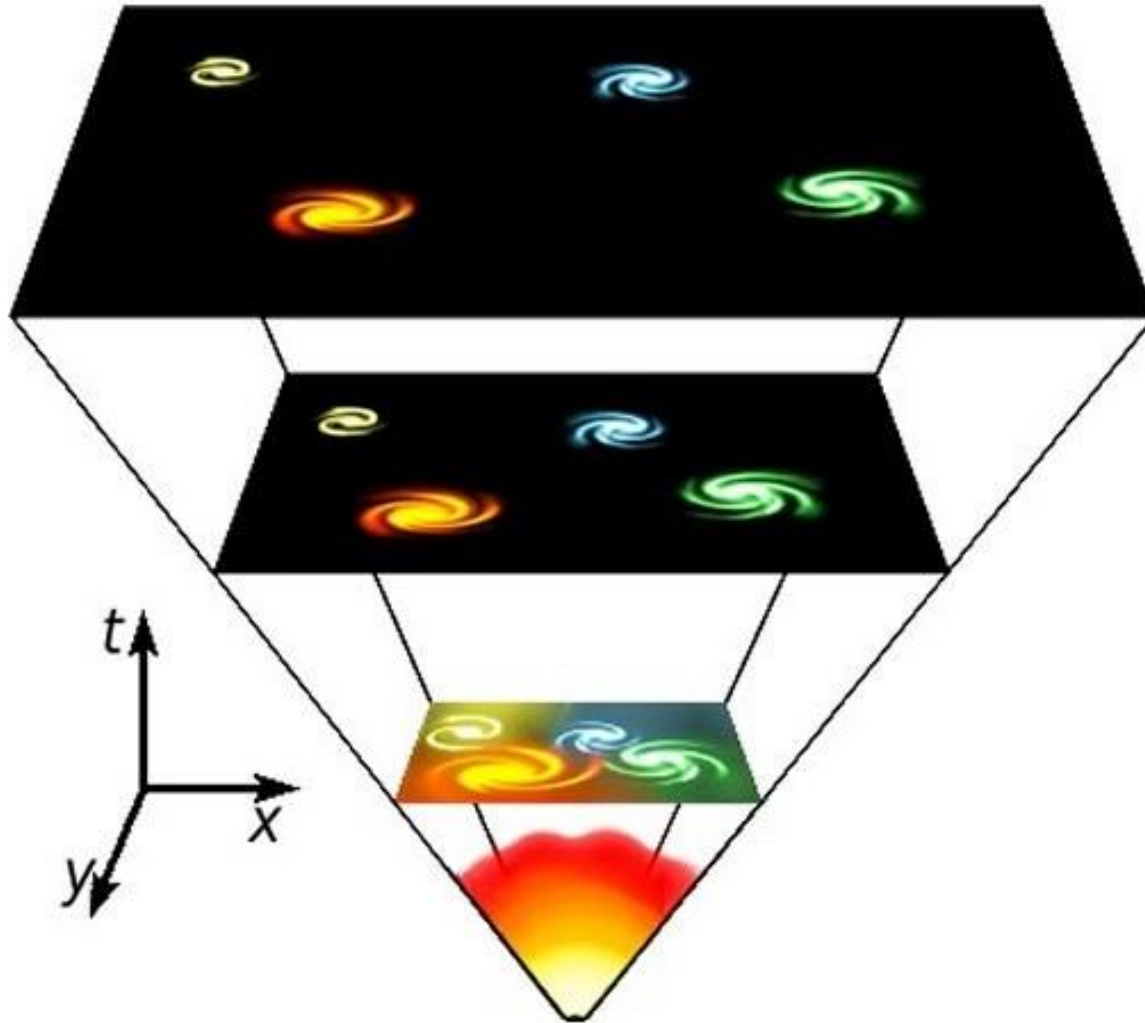


Critical assumption



Move far away from sources:
'spacetime becomes flat'

Expanding spacetimes are not asymptotically flat!



Why assume asymptotic flatness?

P. G. BERGMANN:

The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.

H. BONDI:

I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.

Conference Warsaw 1962

State of the art in cosmology

- Linear perturbation theory
 - Homogeneous solutions \rightarrow CMB
 - Inhomogeneous solutions from compact sources \rightarrow geometric optics approximation

Goal: What can we learn from the full non-linear theory in this setting?

Expansion rates

Expanding
spacetimes

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graph LR; A[Expanding spacetimes] --- B[Acceleration]; A --- C[Deceleration];
```

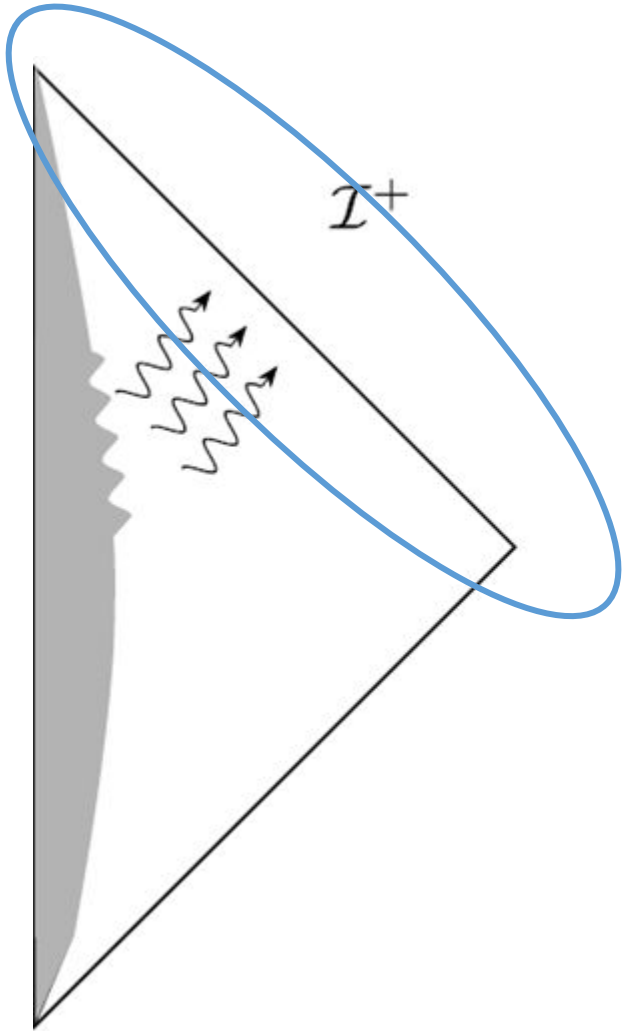
Acceleration

- * Current universe
- * $\Lambda > 0$

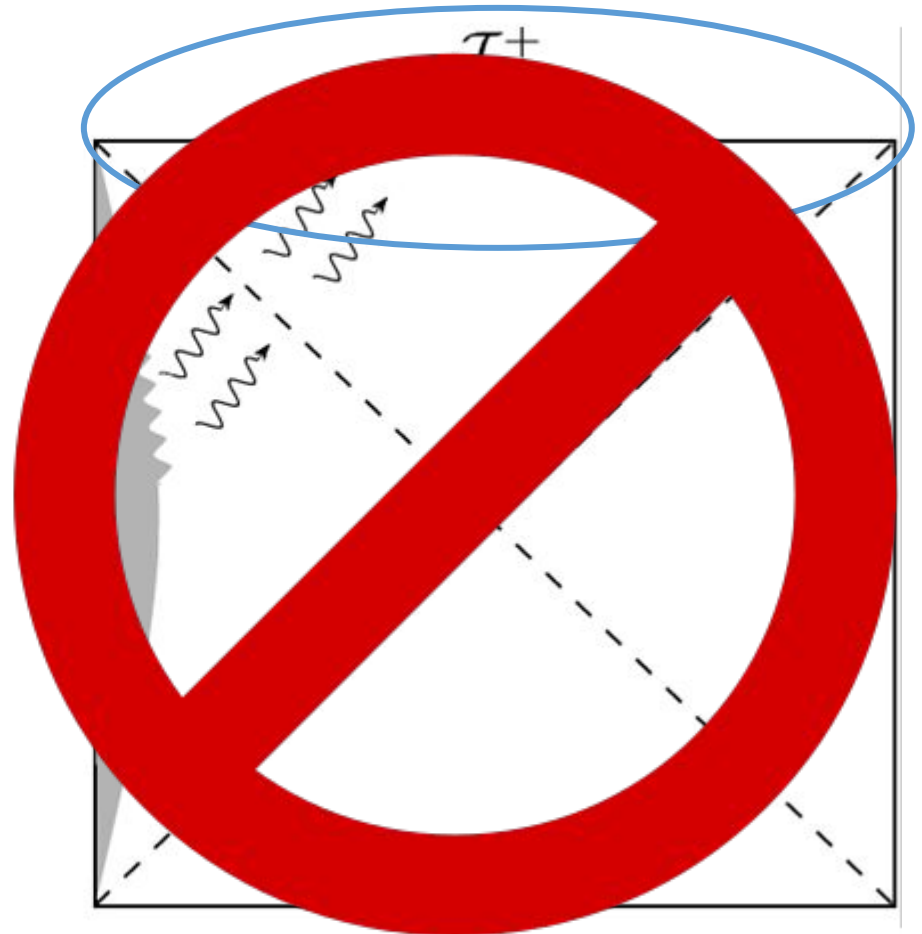
Deceleration

- * Matter dominated era
- * Radiation dominated era
- * $\Lambda = 0$

Radiation zones



$\Lambda = 0$



$\Lambda > 0$

Decelerating FLRW spacetimes

$$d\hat{s}^2 = a^2(\eta) [-d\eta^2 + dr^2 + r^2 \sum_{AB} dx^A dx^B]$$

physical
metric

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{\frac{2}{1-s}}$$

$$s = \frac{2}{3(1+W)}$$

$$0 \leq s < 1$$

$$-1/3 < W < \infty$$

$$\underline{\underline{P = w\rho}}$$

$W = 1$ stiff fluid

$W = 1/3$ radiation

$W = 0$ dust

$W = -1$ cosmological
constant

$$d\hat{s}^2 = a^2(\eta) [-d\eta^2 + dr^2 + r^2 \delta_{AB} dx^A dx^B]$$

$$\eta = \frac{\sin T}{\cos R + \cos T}$$

$$= \frac{\sin\left(\frac{V+U}{2}\right)}{2 \cos\frac{U}{2} \cos\frac{V}{2}}$$

$$r = \frac{\sin R}{\cos R + \cos T}$$

$$= \frac{\sin\left(\frac{V-U}{2}\right)}{2 \cos\frac{U}{2} \cos\frac{V}{2}}$$

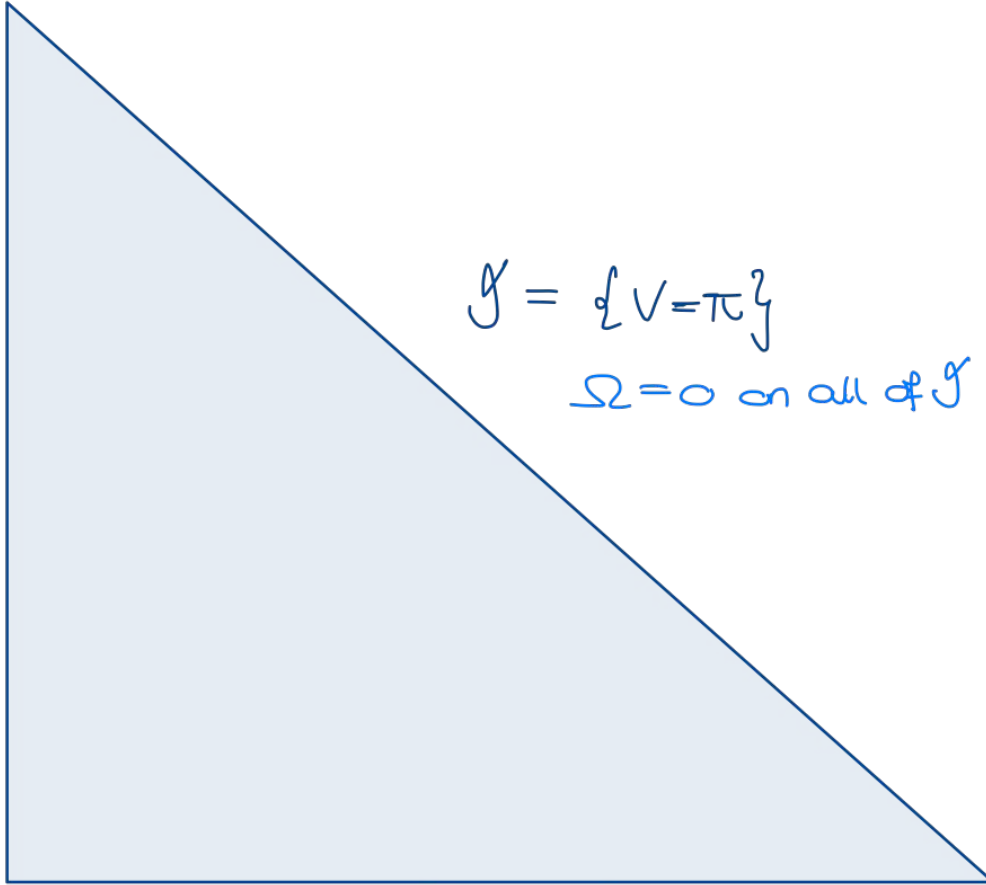
$$\left. \begin{cases} U = T - R \\ V = T + R \end{cases} \right\} \Leftrightarrow \begin{cases} -\pi < U < \pi \\ |U| < V < \pi \end{cases}$$

Choose $\Omega = 2 \left(\cos\frac{U}{2} \cos\frac{V}{2} \right)^{\frac{1}{1-s}} \left(\sin\frac{U+V}{2} \right)^{\frac{-s}{1-s}}$

$$ds^2 = \Omega^2 d\hat{s}^2 = -dU dV + \sin\left(\frac{V-U}{2}\right)^2 \delta_{AB} dx^A dx^B$$

→ Can add $V = -U$ & $V = \pi$, because this metric is smooth everywhere including at the boundaries

$$i^+ = \{V = U = \pi\}$$



$$\mathcal{G} = \{V = \pi\}$$

$\Omega = 0$ on all of \mathcal{G}

Big Bang = $\{V = -U\}$
 Ω diverges here

$$i^0 = \{V = -U = \pi\}$$

The conformal factor

But near g ,...

$$\Omega \sim \cos \frac{U}{2} (\pi - V)^{\frac{1}{1-s}}$$

→ NOT smooth!

$$\nabla_a \Omega \sim \cos \frac{U}{2} (\pi - V)^{\frac{s}{1-s}} \nabla_a V \rightarrow \hat{=} 0 \text{ unless } s=0$$

Bad choice for Ω ?

What to do?

$$\Omega' = \omega \Omega$$

$$\text{with } \omega \sim (\pi - V)^{-\frac{s}{1-s}}$$

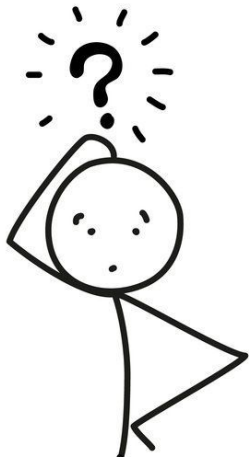
* Ω is smooth @ g ✓

* $\nabla_a \Omega \neq 0$ ✓

but then

$$g'^{ab} = \Omega'^2 g^{ab} \sim (\pi - V)^{\frac{-2s}{1-s}} g^{ab}$$

THIS DIVERGES @ g !



Simple resolution

Ω^{1-s} is smooth @ \mathcal{Y} 😊

* $\Omega^{1-s} \hat{=} 0$

* $\nabla_a \Omega^{1-s} \neq 0$

Define the normal to \mathcal{Y} using Ω^{1-s}

$$\begin{aligned} \Rightarrow n_a &= \frac{1}{1-s} \nabla_a \Omega^{1-s} \\ &= \Omega^{-s} \nabla_a \Omega \\ &\hat{=} -\frac{2^{-s}}{1-s} \left(\cos \frac{\theta}{2}\right)^{1-s} \nabla_a V \end{aligned}$$

Presence of matter

For asymptotically flat spacetimes, $\Omega^{-2} \hat{T}_{ab}$ should have a limit to \mathcal{I} but FLRW spacetimes are homogeneous, so there is matter *everywhere!*

$$\lim_{\rightarrow \mathcal{I}} 8\pi G g^{ab} \hat{T}_{ab} = \frac{6S(1-S)}{(1-S)^2} \left(\sec \frac{U}{2} \right)^2 \rightarrow \text{NON-VANISHING}$$

$$8\pi G \hat{T}_{ab} = \underbrace{2S \Omega^{2(S-1)} n_a n_b}_{\text{universal}} + 2S \Omega^{S-1} T_{(a} n_{b)} + \text{finite}$$

depends on choice Ω
 $T_a \hat{=} \tan \frac{U}{2} (\nabla_a U + \nabla_a V)$

Spacetimes with a cosmological null asymptote

A physical spacetime (\hat{M}, \hat{g}_{ab}) admits a cosmological null asymptote if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{I} \cong \mathbb{R} \times \mathbb{S}^2$ such that

(1) $\Omega \hat{=} 0$, Ω^{1-s} and $g_{ab} = \Omega^2 \hat{g}_{ab}$ is smooth on M ,
 $n_a = \Omega^{-s} \nabla_a \Omega$ is nowhere vanishing on \mathcal{I} (for $0 \leq s < 1$)

(2) Einstein's equations are satisfied with \hat{T}_{ab} such that

$$\lim_{\rightarrow \mathcal{I}} g^{ab} \hat{T}_{ab} \quad \text{exists}$$

$$\lim_{\rightarrow \mathcal{I}} \Omega^{1-s} \left[8\pi \hat{T}_{ab} - 2s \Omega^{2(s-1)} n_a n_b \right] \hat{=} 2s \tau_{(a} n_{b)}$$

Spacetimes with a cosmological null asymptote

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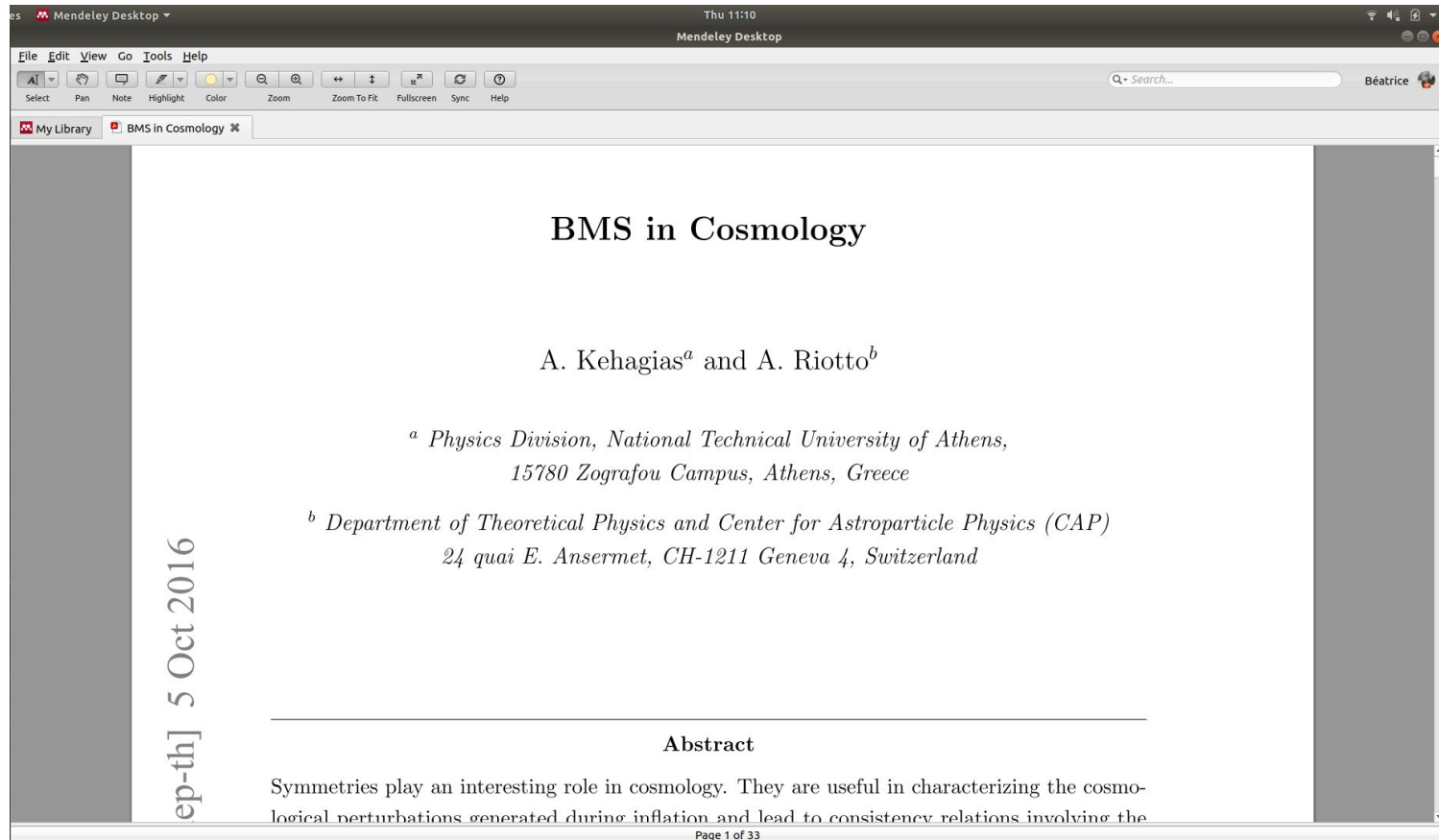
Asymptotic symmetry algebra

All smooth vector fields that map

$$\{q_{ab}, n^a\} \longrightarrow \{q'_{ab} = \omega^2 q_{ab}, n'^a = \omega^{-1-s} n^a\}$$

$$\implies \mathfrak{b}_s \cong \mathfrak{so}(1,3) \ltimes \mathcal{S}_s$$

Didn't we know this already?



The image shows a screenshot of a Mendeley Desktop application window. The window title is "Mendeley Desktop" and the system clock shows "Thu 11:10". The application menu bar includes "File", "Edit", "View", "Go", "Tools", and "Help". Below the menu bar is a toolbar with icons for "Select", "Pan", "Note", "Highlight", "Color", "Zoom", "Zoom To Fit", "Fullscreen", "Sync", and "Help". A search bar on the right contains the text "Search..." and the name "Béatrice". The main content area displays a preprint page for "BMS in Cosmology" by A. Kehagias^a and A. Riotto^b. The authors' affiliations are listed: ^a Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece; and ^b Department of Theoretical Physics and Center for Astroparticle Physics (CAP), 24 quai E. Ansermet, CH-1211 Geneva 4, Switzerland. The page is labeled "ep-th] 5 Oct 2016" on the left side. The abstract section is titled "Abstract" and begins with "Symmetries play an interesting role in cosmology. They are useful in characterizing the cosmological perturbations generated during inflation and lead to consistency relations involving the". The page number "Page 1 of 33" is visible at the bottom.

File Edit View Go Tools Help

Select Pan Note Highlight Color Zoom Zoom To Fit Fullscreen Sync Help

Q+ Search... Béatrice

My Library BMS in Cosmology

BMS in Cosmology

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ep-th] 5 Oct 2016

Abstract

Symmetries play an interesting role in cosmology. They are useful in characterizing the cosmological perturbations generated during inflation and lead to consistency relations involving the

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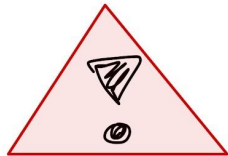
There is a twist!

A supertranslation can again be written as

$$\xi^a \partial_a = \underbrace{f(\theta, \varphi)}_{\text{has conformal weight } 1+S} \partial_u$$

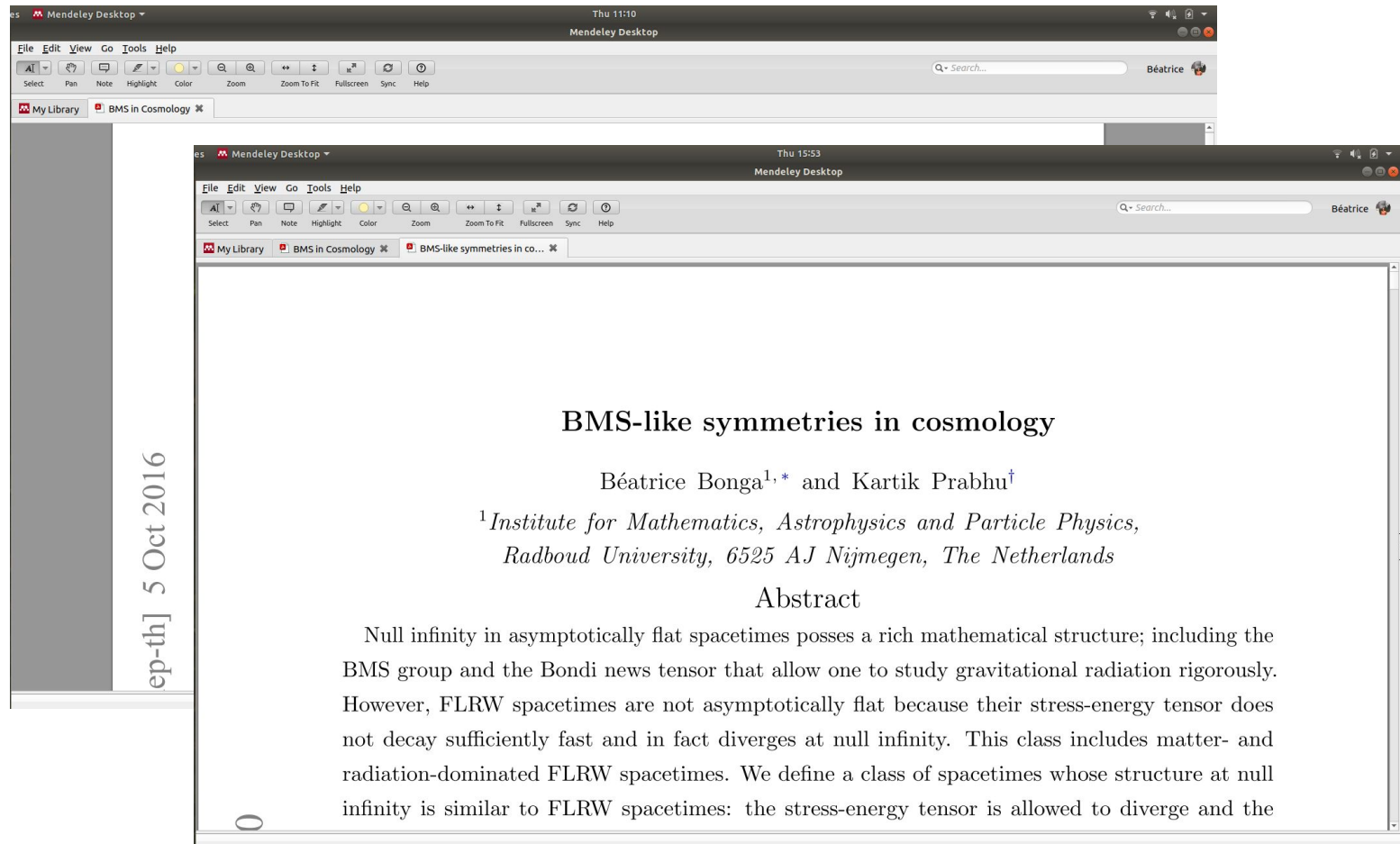
has conformal weight $1+S$

Who cares?



No translation subalgebra

Not exactly BMS in cosmology



ep-th] 5 Oct 2016

BMS-like symmetries in cosmology

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Abstract

Null infinity in asymptotically flat spacetimes possesses a rich mathematical structure; including the BMS group and the Bondi news tensor that allow one to study gravitational radiation rigorously. However, FLRW spacetimes are not asymptotically flat because their stress-energy tensor does not decay sufficiently fast and in fact diverges at null infinity. This class includes matter- and radiation-dominated FLRW spacetimes. We define a class of spacetimes whose structure at null infinity is similar to FLRW spacetimes: the stress-energy tensor is allowed to diverge and the



notion of mass and linear momentum?

Challenge for extracting radiation

$$\Psi_4 := -C_{abcd}\bar{m}^a n^b \bar{m}^c n^d \hat{=} 0, \quad \tilde{\Omega}^{-1}\Psi_4 \hat{=} \left(\frac{1}{2}\partial_u^2 C_{AB} + s\partial_u \bar{\delta}_{A\tau B} + \frac{1}{2}s^2\tau_A\partial_u\tau_B\right)\bar{m}^A\bar{m}^B$$

$$\Psi_3 := -C_{abcd}l^a n^b \bar{m}^c n^d \hat{=} -\frac{s}{4}\partial_u\tau_A\bar{m}^A$$

$$\Psi_2 := -C_{abcd}l^a m^b \bar{m}^c n^d \hat{=} -\frac{1}{6}\left[W^{(2)} - 1 - s\left(\partial_u\tau + \frac{1}{2}\bar{\delta}_{A\tau}{}^A + s\tau_A\tau^A + \frac{3}{2}i\epsilon^{AB}\bar{\delta}_{A\tau B}\right)\right]$$



No peeling!

Any other examples?

Class of spacetimes at least as big as asymptotically flat spacetimes!



Linearization stability still open question!

Conclusion

- ❖ Gravitational radiation can be studied using asymptotics in the full non-linear theory
 - ❖ Asymptotic symmetry algebra provides charges and fluxes with a physical interpretation
 - Asymptotic flat spacetimes: BMS
 - Asymptotic cosmological null asymptotes: BMS-*like*, there is no translation subalgebra!
- + *ongoing work with Berend Schneider & Sk Jahanur Hoque*