Foundations of

Gravitational waves in cosmology

Béatrice Bonga – 4 Feb 2022 – Second Chennai Symposium on Gravitation and Cosmology

[BB+Prabhu, arXiv:2009.01243]

Radboud University

Invaluable tool: perturbation theory

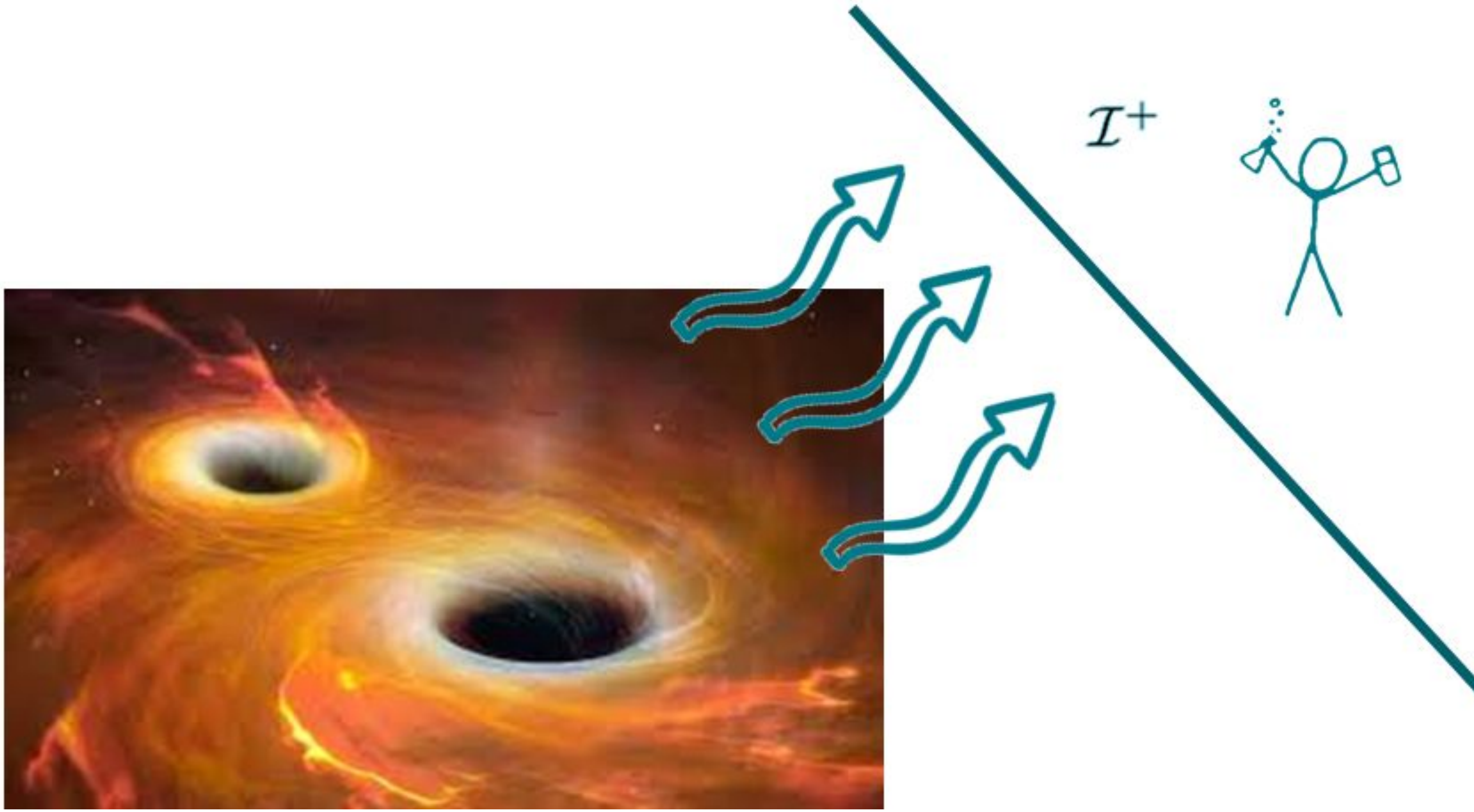
$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h_{\mu\nu}$$

Minkowski spacetime
Kerr spacetime
FLRW spacetime
⋮

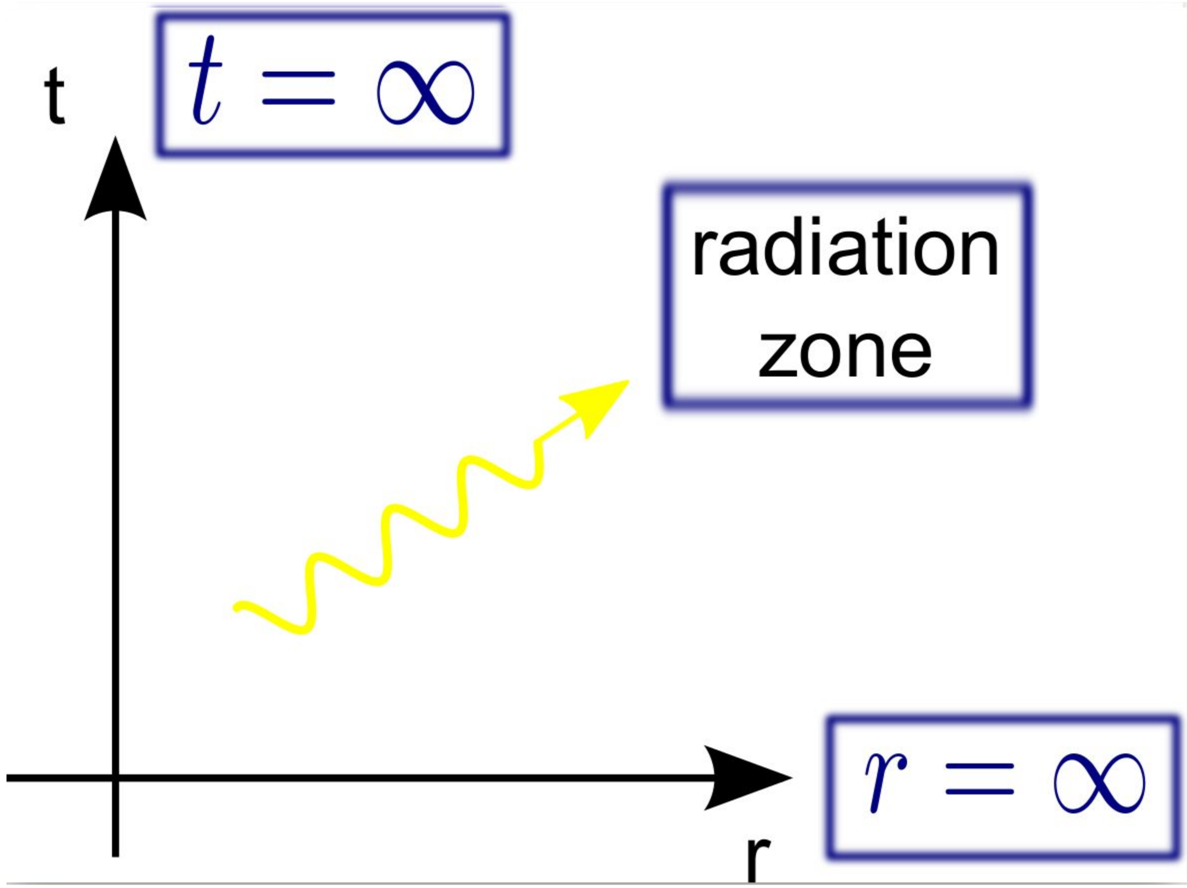
But there is no canonical split!

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h_{\mu\nu}$$

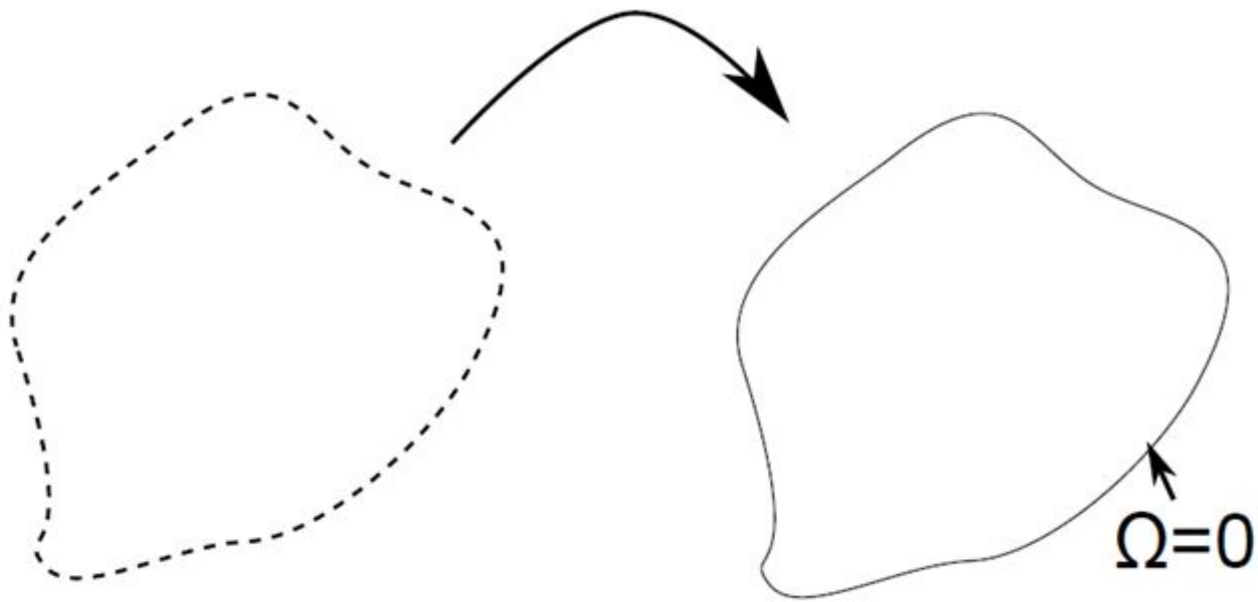
From messy physics to peaceful realm



Different infinities



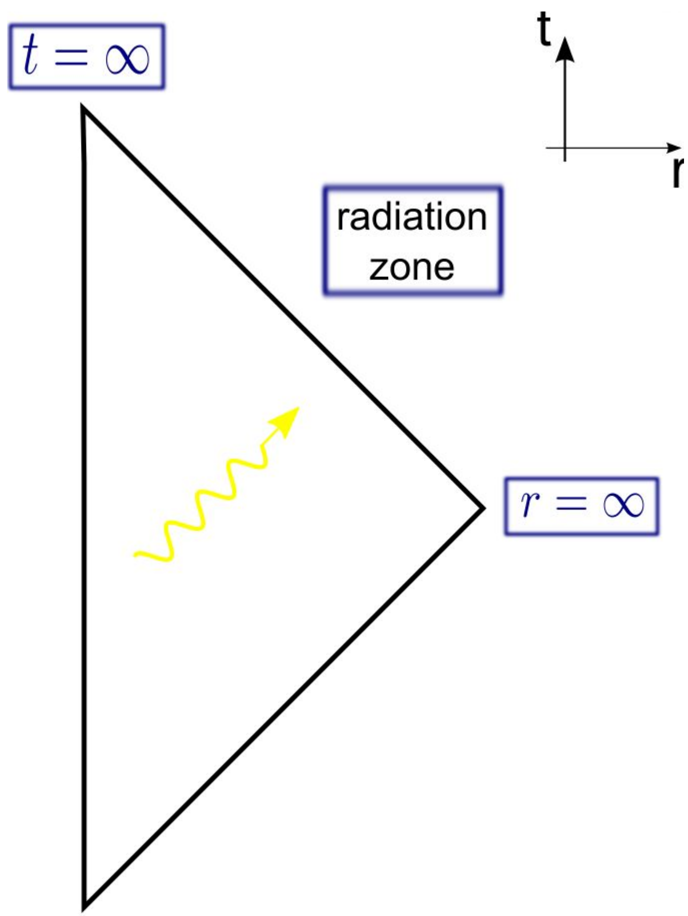
Key idea: bring infinity to a finite distance



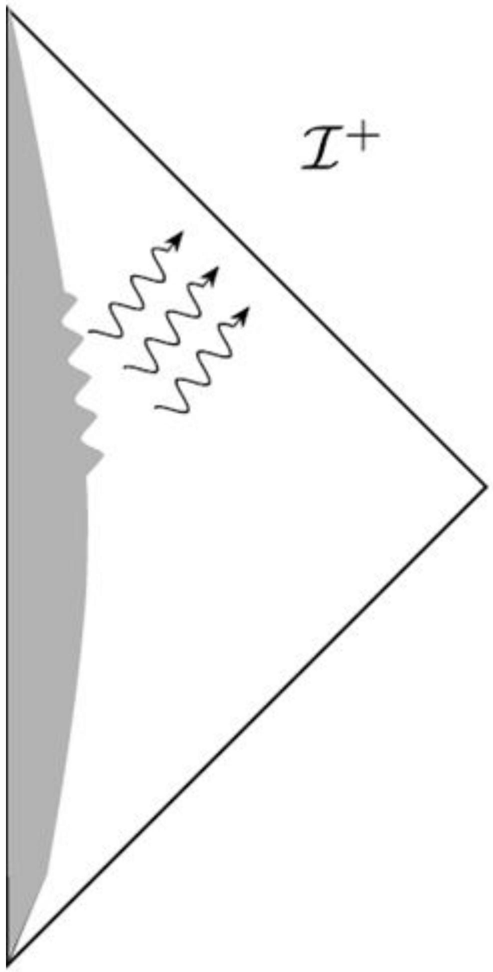
$(\hat{M}, \hat{g}_{ab})$ $\quad$ $(M, g_{ab} = \Omega^2 \hat{g}_{ab})$

Conformal completion

Conformal diagram Minkowski



Asymptotic flatness

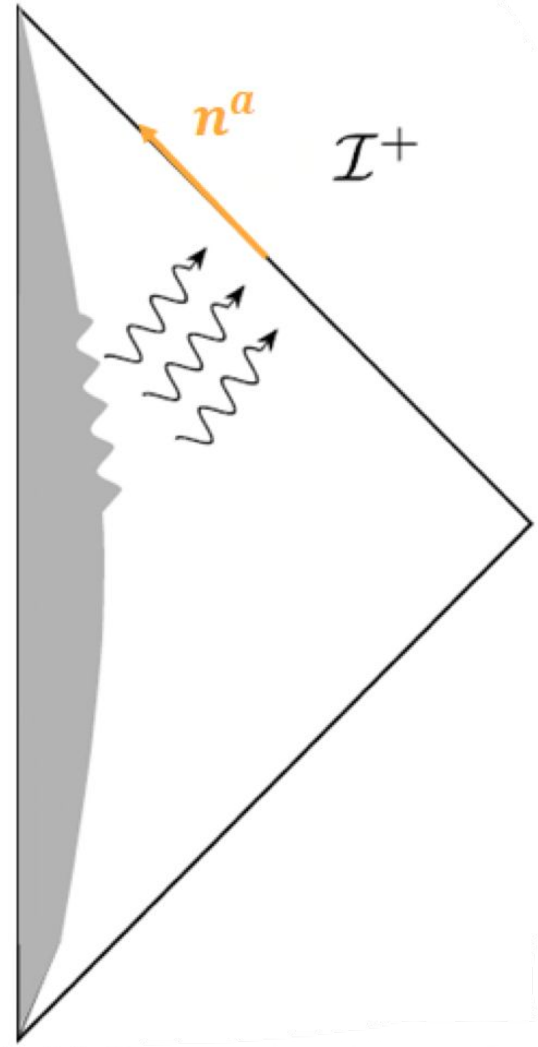


A physical spacetime $(\hat{M}, \hat{g}_{ab})$ is asymptotically flat if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{J} \cong \mathbb{R}\, x\, \mathbb{S}^2$ such that

1. Ω and $g_{ab} = \Omega^2 \ \hat{g}_{ab}$ are smooth on M, $\Omega \ \hat{=}\ 0$ and $n_a = \nabla_a \Omega$ is nowhere vanishing on $\mathcal{J}$
2. Einstein's equations are satisfied with $\hat{T}_{ab}$ such that $\Omega^{-2}\hat{T}_{ab}$ has a smooth limit to $\mathcal{J}$

Consequences

- Einstein's equation ⟹ n^a is null on $\mathscr{I}$
- q_{ab} = induced metric on $\mathscr{I}$ is degenerate: 0 + +



Universal structure

This is common to ***all*** asymptotically flat spacetimes

$$\{ q_{ab}, n^a \} = \{ \omega^2 q_{ab}, \omega^{-1} n^a \}$$

Gravitational radiation is encoded in the next-order structure and differs from spacetime to spacetime

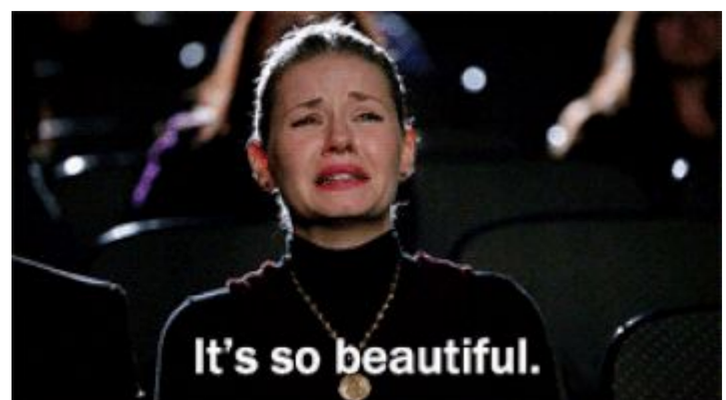
Key points of asymptotics

Nowhere in this construction did we introduce a split of the background and "gravitational waves".

The split occurs naturally at null infinity:

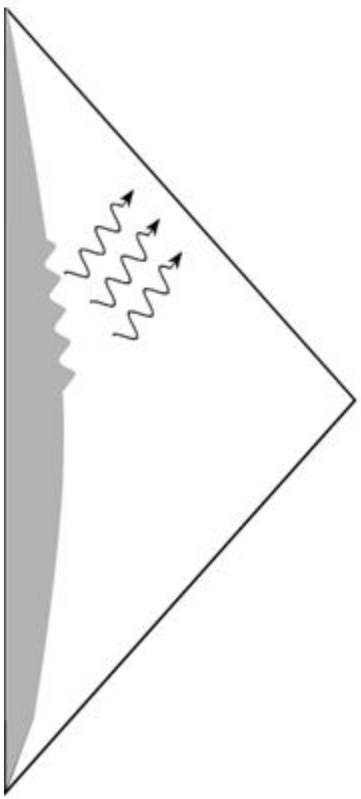
- universal structure is like a background,
- first order structure contains gravitational radiation,

and it is fully non-linear!



Generic asymptotically flat spacetime

$$d\hat{s}^2 = -UV\, du^2 - 2U\, dudr + \gamma_{AB}(r\, d\theta^A + W^A\, du)(r\, d\theta^B + W^B\, du)$$



$$U = 1 + B/r^2 + O(r^{-3}),$$
$$V = 1 - 2M/r + N/r^2 + O(r^{-3}),$$
$$W^A = A^A/r + B^A/r^2 + O(r^{-3}),$$
$$\gamma_{AB} = \Omega_{AB} + f_{AB}/r + \tfrac{1}{4} f^2 \Omega_{AB}/r^2 + O(r^{-3})$$

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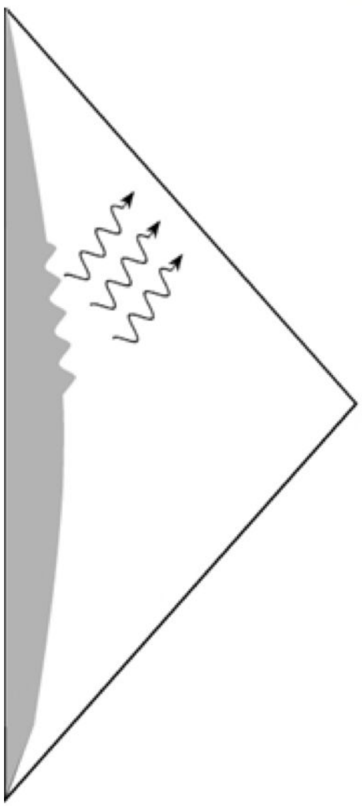

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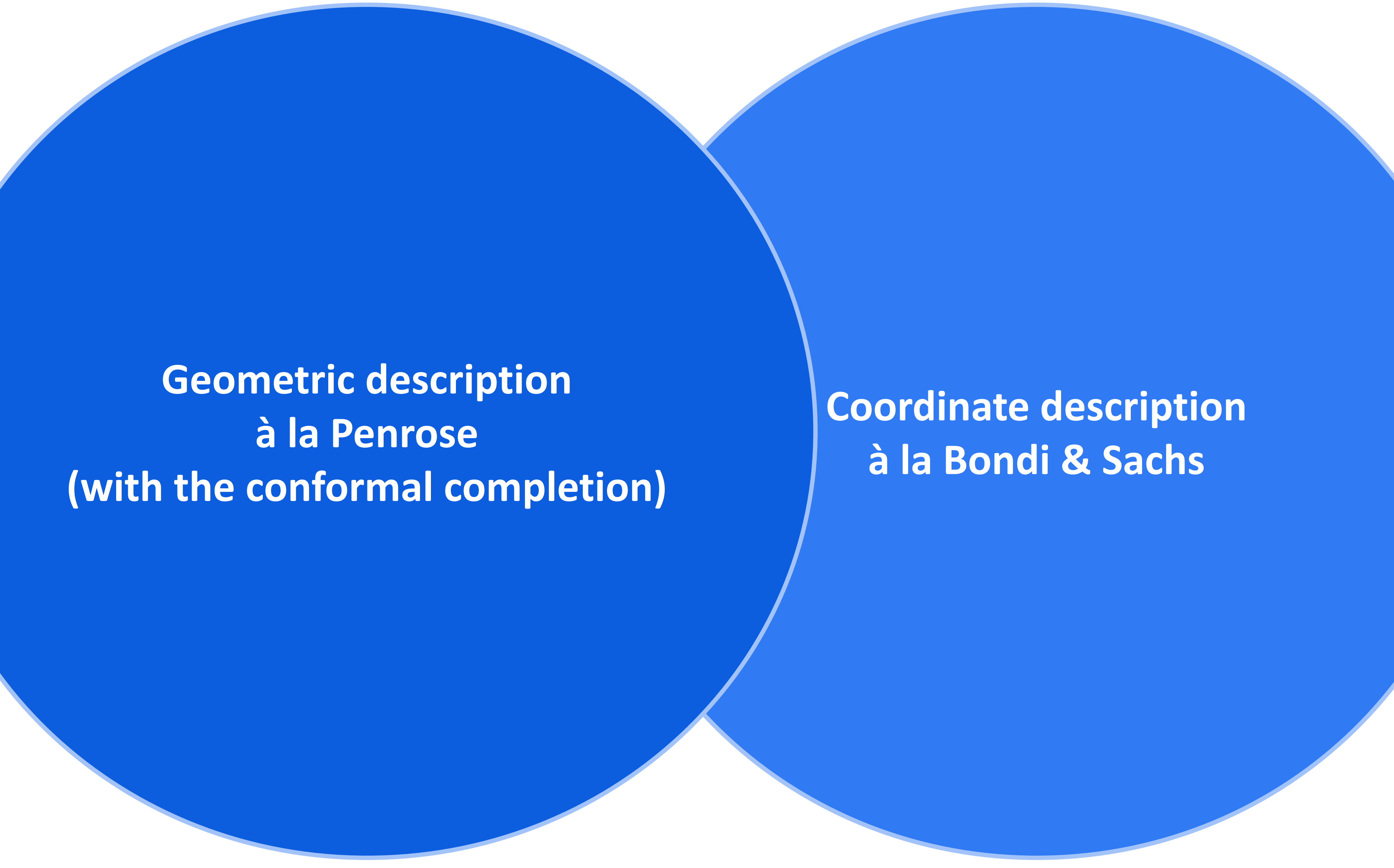
Flat Space | Mass aspect | Radiative modes | Angular momentum aspect

Two definitions are equivalent!



Asymptotic symmetry algebra

Spacetime diffeomorphism that leave the universal structure at null infinity invariant

Universal structure

$$\mathcal{L}_\xi q_{ab} \mathrel{\widehat{=}} 2\,\alpha\, q_{ab} \text{ with } \mathcal{L}_n \alpha \mathrel{\widehat{=}} 0$$

$$\mathcal{L}_\xi n^a \mathrel{\widehat{=}} -\,\alpha\, n^a$$

Coordinates

$$\Omega^2 \mathcal{L}_\xi \hat{g}_{ab} \mathrel{\widehat{=}} 0$$

Bondi-Metzner-Sachs algebra (BMS)

- Asymptotic symmetry algebra is ***bigger*** than Poincaré
- BMS = supertranslations & rotations

$$\xi^a \partial_a = \left(f(\theta, \varphi) + \frac{1}{2} \, u \, D_A Y^A \right) \partial_u + Y^A \partial_A$$

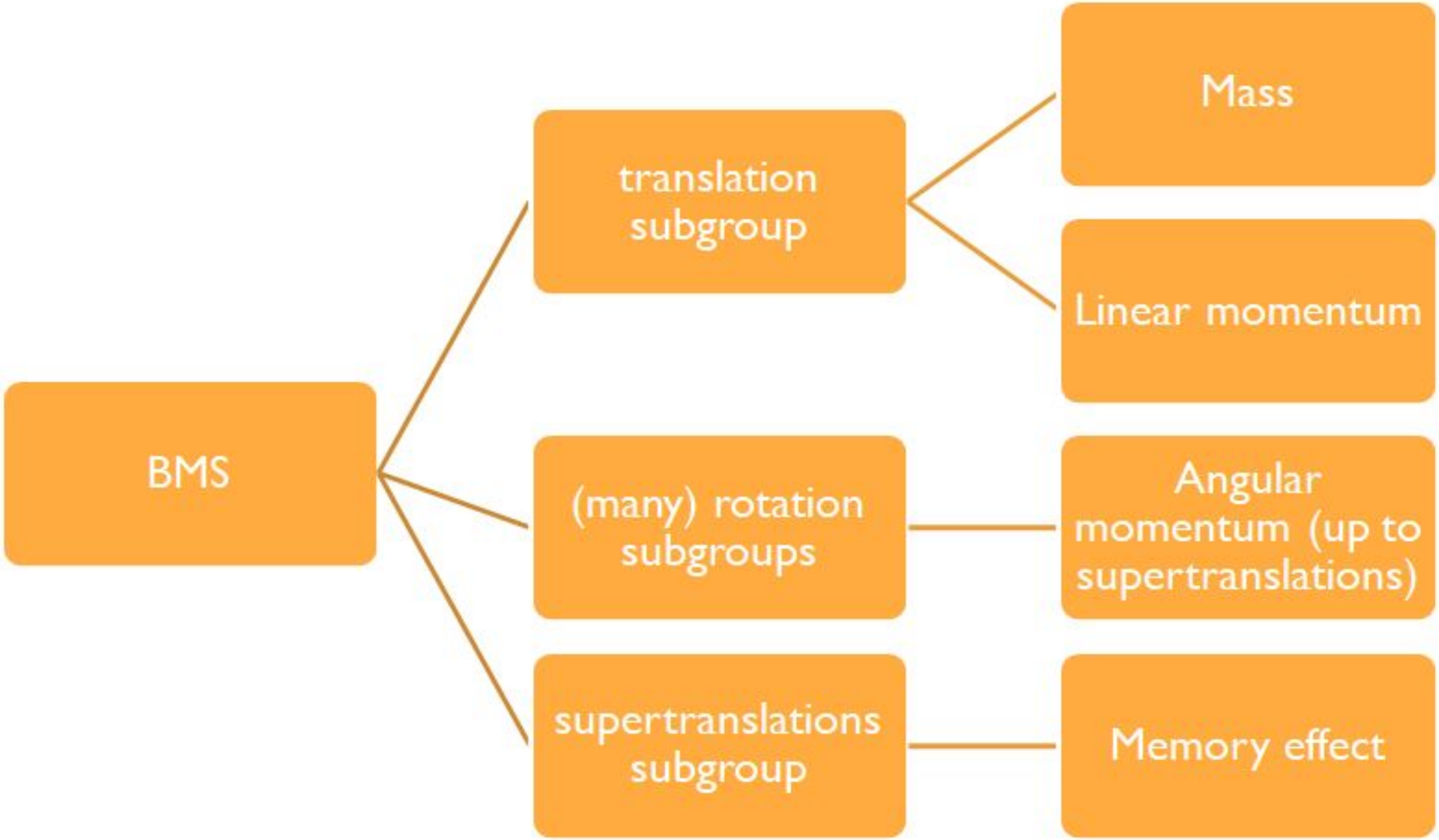
supertranslations

rotations

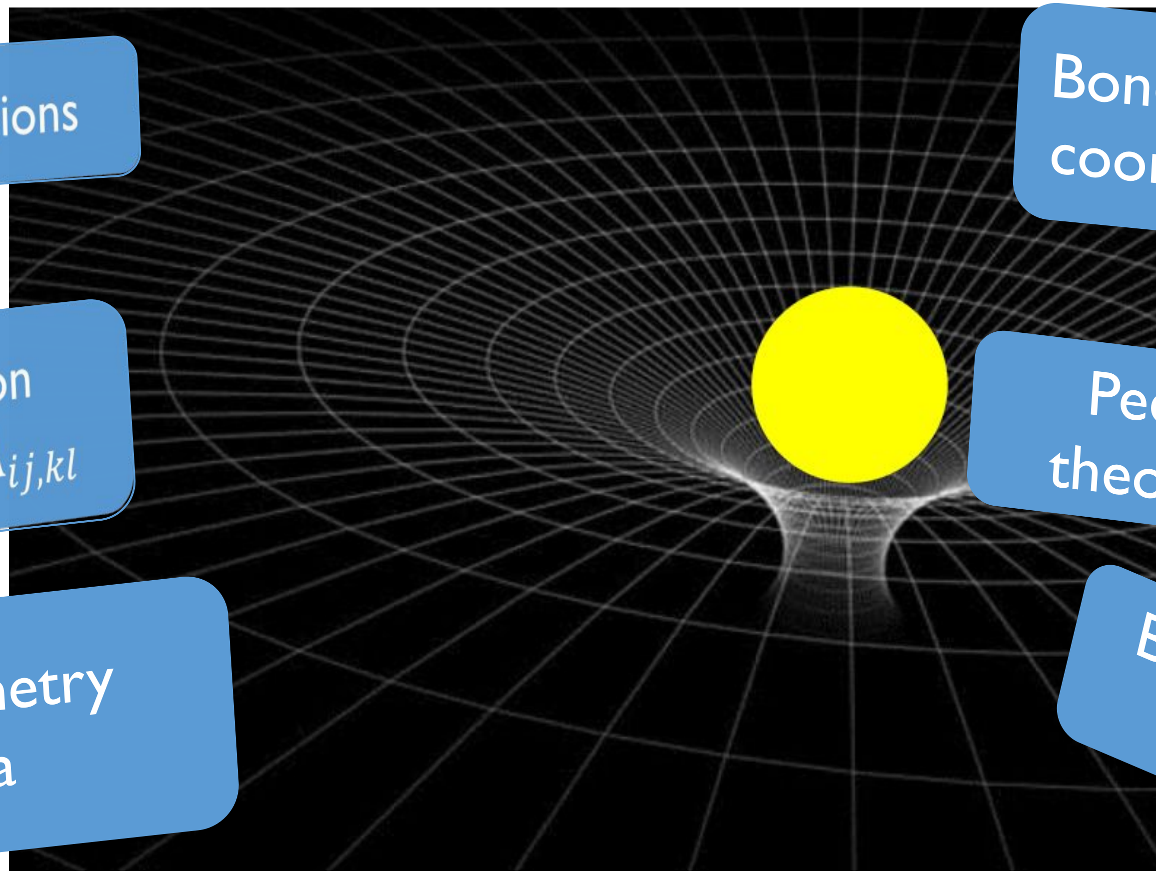
$$2D_{(A}Y_{B)} + q_{AB} D_C Y^C = 0$$

What is BMS good for?

It provides quantities with a physical interpretation!



Critical assumption



$\frac{1}{r}$ expansions

Projection operator $\Lambda_{ij,kl}$

BMS symmetry algebra

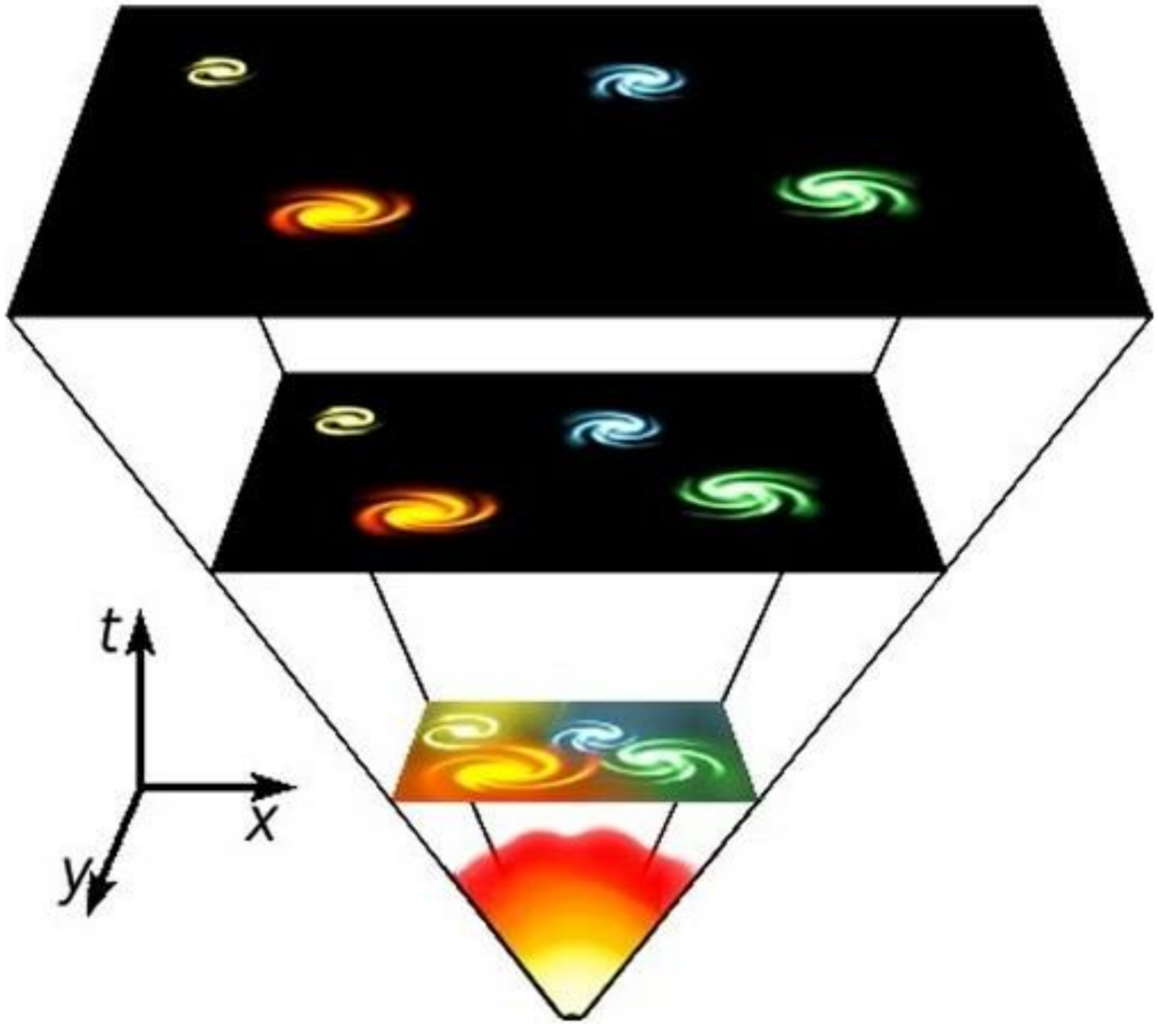
Bondi-Sachs coordinates

Peeling theorems

Bondi news tensor

Move far away from sources:
'spacetime becomes flat'

Expanding spacetimes are not asymptotically flat!



Why assume asymptotic flatness?

P. G. BERGMANN:

The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.

H. BONDI:

I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.

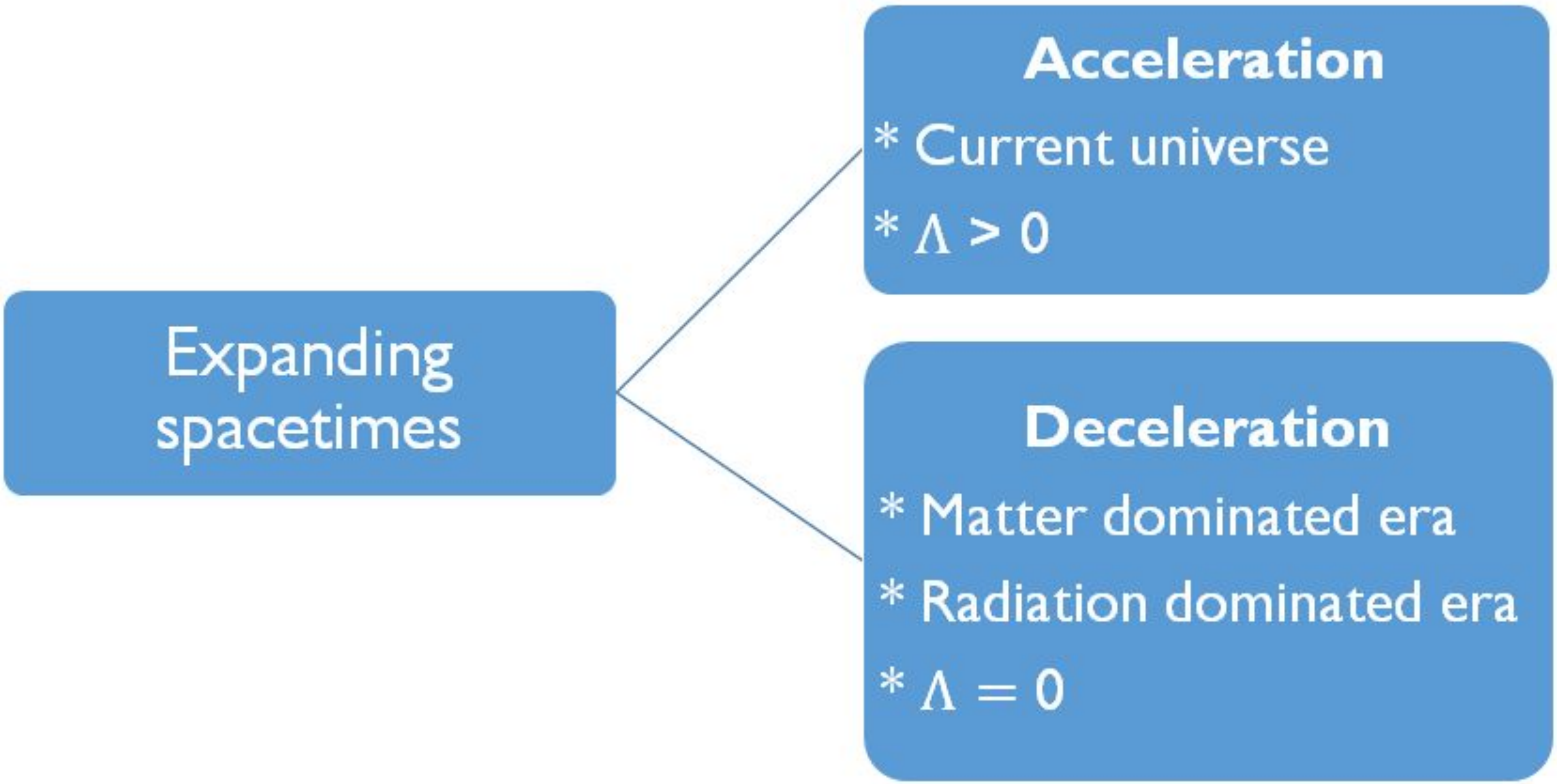
Conference Warsaw 1962

State of the art in cosmology

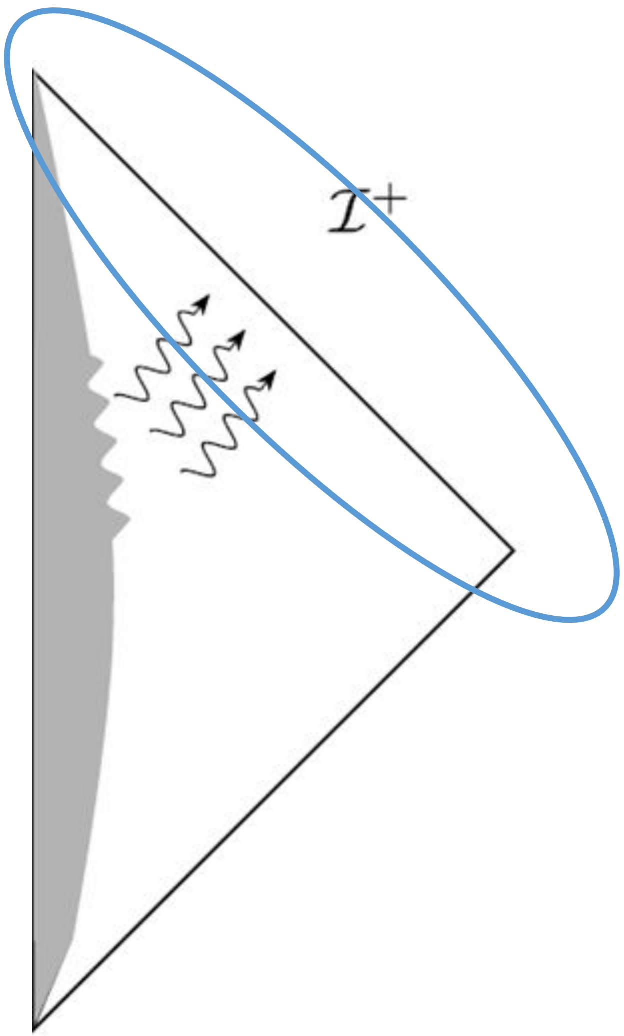
- Linear perturbation theory
  - Homogeneous solutions → CMB
  - Inhomogeneous solutions from compact sources → geometric optics approximation

Goal: What can we learn from the full non-linear theory in this setting?

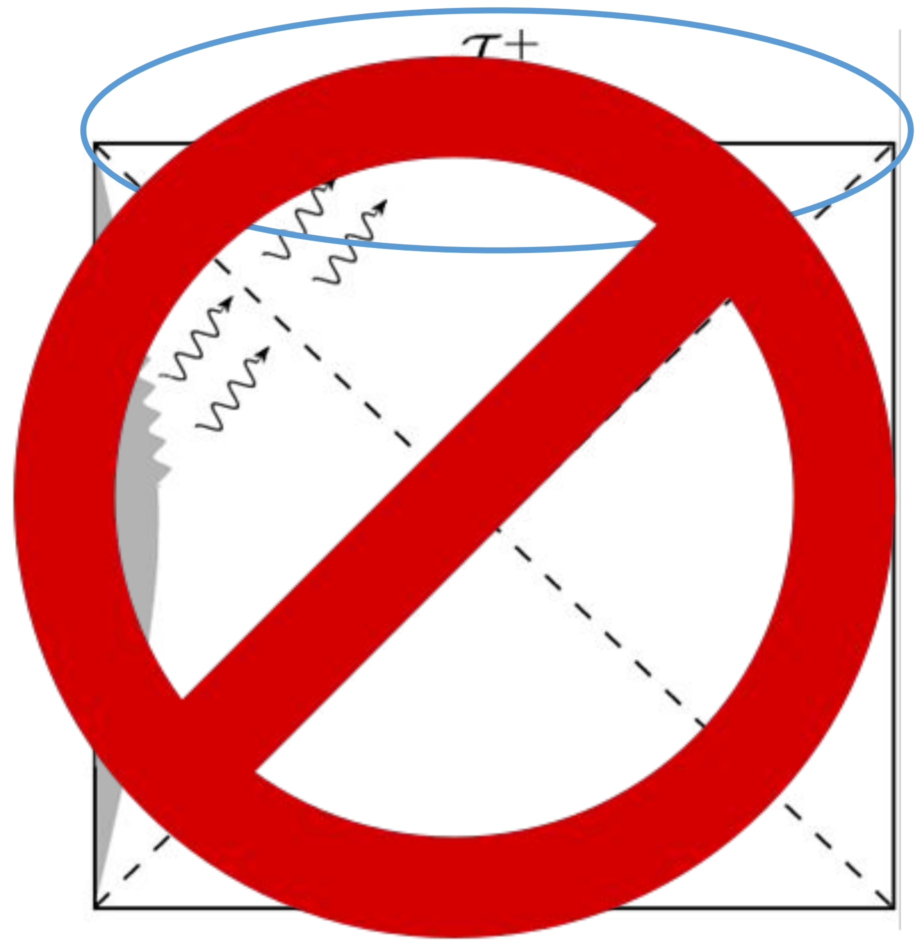
Expansion rates



Radiation zones



$\Lambda = 0$



$\Lambda > 0$

Decelerating FLRW spacetimes

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 \delta_{AB}\, dx^A dx^B\right]$$

physical metric

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{\frac{s}{1-s}}$$

$$s = \frac{2}{3(1+w)}$$

$$0 \leq s < 1$$

$$-1/3 < w < \infty$$

$$P = w\rho$$

$w = 1$	stiff fluid
$w = 1/3$	radiation
$w = 0$	dust
$w = -1$	cosmological constant

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 \delta_{AB}\, dx^A dx^B\right]$$

$$\eta = \frac{\sin T}{\cos R + \cos T} = \frac{\sin\left(\frac{V+U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

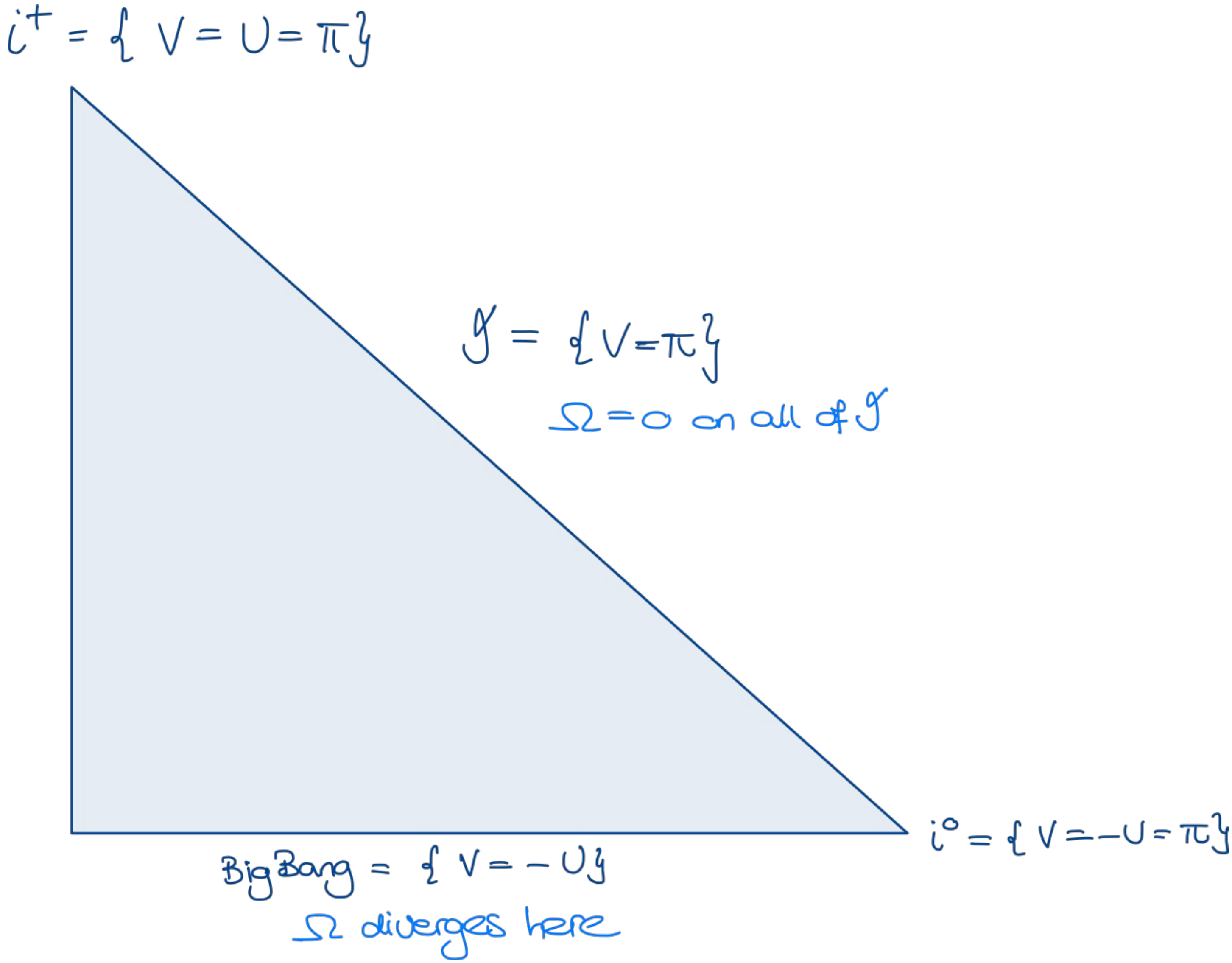
$$r = \frac{\sin R}{\cos R + \cos T} = \frac{\sin\left(\frac{V-U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

$$\begin{cases} U = T - R \\ V = T + R \end{cases} \Longleftrightarrow \begin{array}{l} -\pi < U < \pi \\ |U| < V < \pi \end{array}$$

Choose $\Omega = 2\left(\cos\frac{U}{2}\cos\frac{V}{2}\right)^{\frac{1}{1-s}}\left(\sin\frac{U+V}{2}\right)^{-\frac{s}{1-s}}$

$$ds^2 = \Omega^2 d\hat{s}^2 = -\,dU\,dV + \sin\left(\frac{V-U}{2}\right)^2 \delta_{AB}\, dx^A dx^B$$

$\longrightarrow$ can add $V = -U$ & $V = \pi$, because this metric is smooth everywhere including at the boundaries



$i^+ = \{V = U = \pi\}$
$\mathscr{I} = \{V = \pi\}$
$\Omega = 0$ on all of $\mathscr{I}$
$i^0 = \{V = -U = \pi\}$
BigBang $= \{V = -U\}$
Ω diverges here

The conformal factor

But near $\mathscr{I}$...

$$\Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{1}{1-s}} \;\longrightarrow\; \text{NOT smooth!}$$

$$\nabla_a\Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{s}{1-s}}\,\nabla_a V \;\longrightarrow\; \hat{=}\,0 \text{ unless } s=0$$

Bad choice for Ω?

What to do?

$$\Omega' = \omega\,\Omega$$

with $\omega \sim (\pi - V)^{-\frac{s}{1-s}}$ $\Rightarrow$

* Ω is smooth @ $\mathscr{I}$ ☑
* $\nabla_a\Omega \neq 0$ ☑

but then

$$g'_{ab} = \Omega'^2\,\hat{g}_{ab} \sim (\pi - V)^{-\frac{2s}{1-s}}\,g_{ab}$$

THIS DIVERGES @ $\mathscr{I}$!

Simple resolution

Ω^{1-s} is smooth @ $\mathscr{I}$

* $\Omega^{1-s} \hat{=} 0$
* $\nabla_a \Omega^{1-s} \not\hat{=} 0$

Define the normal to $\mathscr{I}$ using Ω^{1-s}

$$
\begin{aligned}
\Rightarrow n_a &:= \frac{1}{1-s} \nabla_a \Omega^{1-s} \\
&= \Omega^{-s} \nabla_a \Omega \\
&\hat{=} -\frac{2^{-s}}{1-s} \left(\cos \frac{U}{2}\right)^{1-s} \nabla_a V
\end{aligned}
$$

Presence of matter

For asymptotically flat spacetimes, $\Omega^{-2}\hat{T}_{ab}$ should have a limit to $\mathcal{I}$ but FLRW spacetimes are homogeneous, so there is matter *everywhere*!

$$\lim_{\to \mathscr{I}} 8\pi G \, g^{ab}\hat{T}_{ab} = \frac{6s(1-s)}{(1-s)^2}\left(\sec\frac{U}{2}\right)^2 \to \text{NON-VANISHING}$$

$$8\pi G \, \hat{T}_{ab} = \underbrace{2s\,\Omega^{2(s-1)}\, n_a n_b}_{\text{universal}} + 2s\,\Omega^{s-1}\, \tau_{(a} n_{b)} + \text{finite}$$

depends on choice Ω

$$\tau_a \,\hat{=}\, \tan\frac{U}{2}\left(\nabla_a U + \nabla_a V\right)$$

Spacetimes with a cosmological null asymptote

A physical spacetime $(\hat{M}, \hat{g}_{ab})$ admits a cosmological null asymptote if there exists a spacetime (M, g_{ab}) with boundary $\partial M \cong \mathcal{J} \cong \mathbb{R} \, x \, \mathbb{S}^2$ such that

(1) $\Omega \;\hat{=}\; 0 \,, \Omega^{1-s}$ and $g_{ab} = \Omega^2 \; \hat{g}_{ab}$ is smooth on M,

$n_a = \Omega^{-s} \nabla_a \Omega$ is nowhere vanishing on $\mathcal{J}$ (for $0 \leq s < 1$)

(2) Einstein's equations are satisfied with $\hat{T}_{ab}$ such that

$\lim_{\to \mathscr{I}} g^{ab} \hat{T}_{ab}$ exists

$$\lim_{\to \mathscr{I}} \Omega^{1-s} \left[8\pi \hat{T}_{ab} - 2s\Omega^{2(s-1)} n_a n_b \right] \hat{=} 2s\tau_{(a} n_{b)}$$

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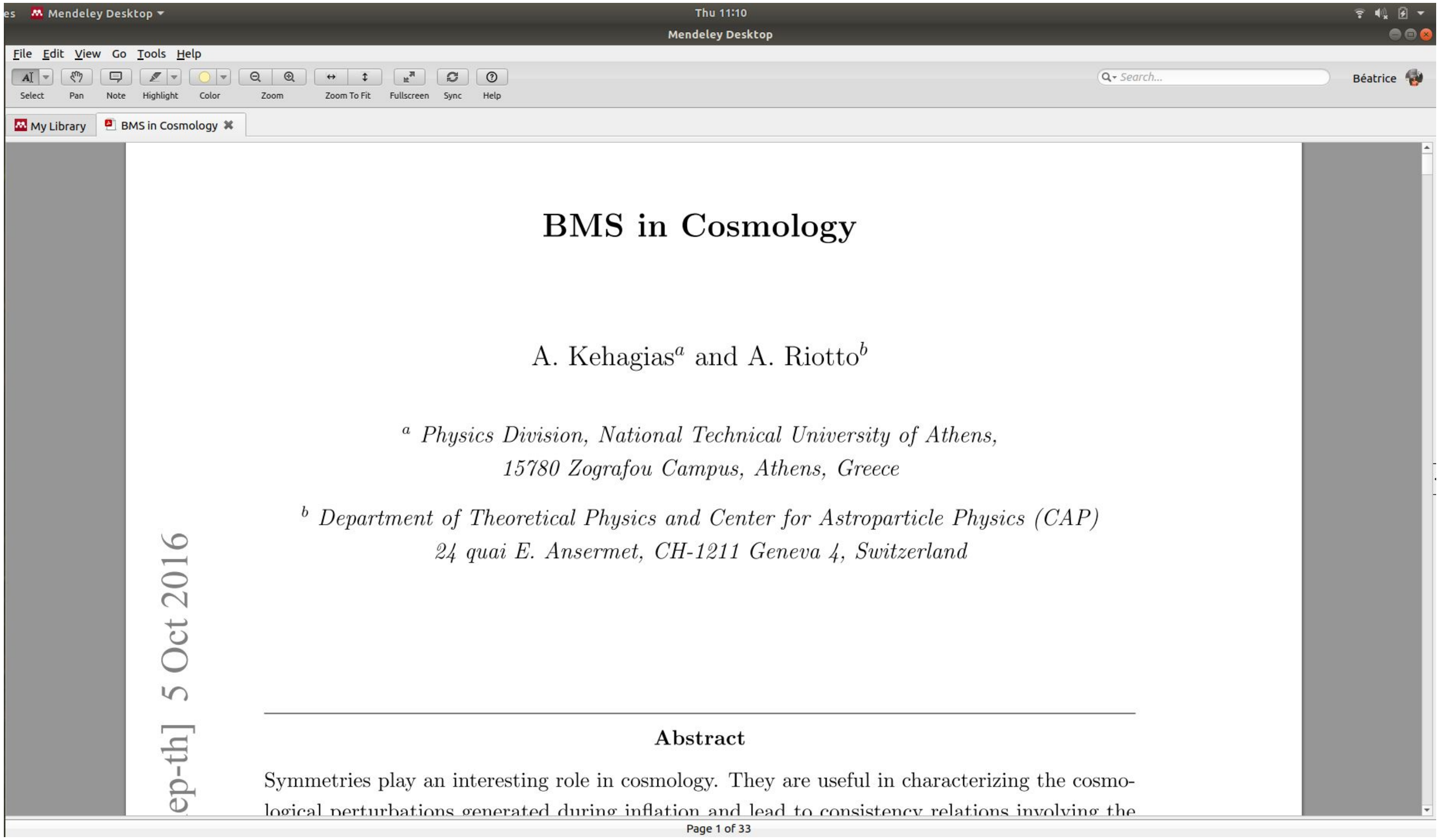
Asymptotic symmetry algebra

All smooth vector fields that map

$$\{q_{ab}, n^a\} \longrightarrow \{q'_{ab} = \omega^2 q_{ab}, n'^a = \omega^{-1-s} n^a\}$$

$$\Longrightarrow \mathfrak{b}_s \cong \mathfrak{so}(1,3) \ltimes \mathfrak{S}_s$$

Didn't we know this already?

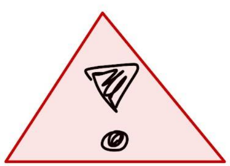


There is a twist!

A supertranslation can again be written as

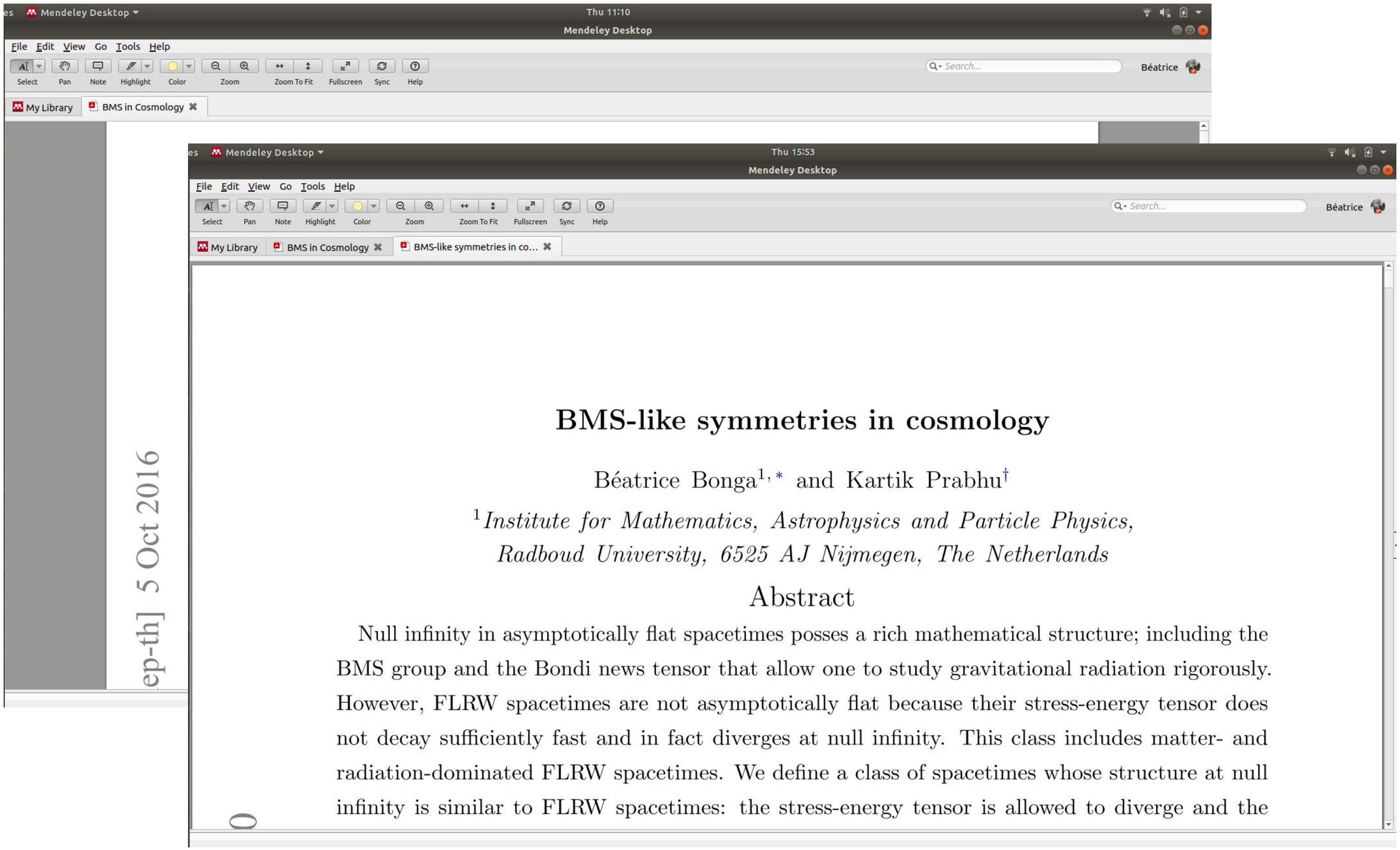
$$\xi^a \partial_a = \underbrace{f(\theta,\varphi)}_{\text{has conformal weight } 1+s} \partial_u$$

Who cares?



No translation subalgebra

Not exactly BMS in cosmology



notion of mass and linear momentum?

Challenge for extracting radiation

$$\Psi_4 := -C_{abcd}\overline{m}^a n^b \overline{m}^c n^d \mathrel{\hat{=}} 0\,, \quad \tilde{\Omega}^{-1}\Psi_4 \mathrel{\hat{=}} \left(\tfrac{1}{2}\partial_u^2 C_{AB} + s\partial_u \eth_A \tau_B + \tfrac{1}{2}s^2\tau_A\partial_u\tau_B\right)\overline{m}^A\overline{m}^B$$

$$\Psi_3 := -C_{abcd}l^a n^b \overline{m}^c n^d \mathrel{\hat{=}} -\tfrac{s}{4}\partial_u\tau_A\overline{m}^A$$

$$\Psi_2 := -C_{abcd}l^a m^b \overline{m}^c n^d \mathrel{\hat{=}} -\tfrac{1}{6}\left[W^{(2)} - 1 - s\left(\partial_u\tau + \tfrac{1}{2}\eth_A\tau^A + s\tau_A\tau^A + \tfrac{3}{2}i\epsilon^{AB}\eth_A\tau_B\right)\right]$$

No peeling!

Any other examples?

Class of spacetimes at least as big as asymptotically flat spacetimes!

Linearization stability still open question!

Conclusion

- ❖ Gravitational radiation can be studied using asymptotics in the full non-linear theory

- ❖ Asymptotic symmetry algebra provides charges and fluxes with a physical interpretation

  - ➢ Asymptotic flat spacetimes: BMS
  - ➢ Asymptotic cosmological null asymptotes: BMS-***like***, there is no translation subalgebra!

+ *ongoing work with Berend Schneider & Sk Jahanur Hoque*