Supertranslations at Timelike infinity

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Outline

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Motivation

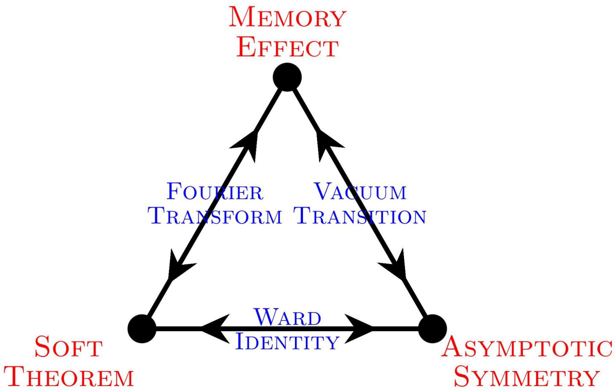

Figure: Infrared Triangle of Strominger

Motivation

- The Infrared triangle gives an equivalence relation that governs the infrared dynamics of all physical theories with massless particles.
- The three vertices denote Soft theorems, Memory effects, and Asymptotic symmetries.
- These ideas are connected by mathematical relations.
- We will only discuss about the vertex representing Asymptotic symmetries.

Introduction

- The asymptotic properties of gravity have been studied extensively for decades in the context of asymptotically flat spacetimes at null infinity.

- The BMS group is a semi direct product of supertranslation group and Lorentz group under which one asymptotically flat solution of general relativity at null infinity is mapped into another solution. [Bondi, Burg and Metzner, Sachs]

- The infinite-dimensional supertranslation subgroup generate arbitrary angle dependent translations of retarded time.

Introduction

- BMS symmtries are present at spatial and timelike infinity, subjected to boundary conditions compatibility

- A natural question to ask is: How do we relate boundary conditions at null, spatial, and timelike infinity? This is difficult.

Our work

- Henneaux and Troessaert in a series of paper have proposed boundary conditions at spatial infinity that are invariant under BMS symmetries. Henneaux and Troessaert [2017-19]
- Our motivation lies in exploring the boundary conditions at timelike infinity that realise BMS symmetries in the sense that it has a non-trivial action and have generically non-zero charges.

Motivation

- Recent studies suggest that stationary black holes also possess an infinite number of symmetries in the near horizon region. [Hawking, Perry, Strominger; Carlip; Donnay et al]

- It is believed that global charges associated with supertranslations receive contributions from the horizon as well as from null infinity.

- Thus, for a complete study of conservation laws associated with supertranslations, it is required to know the relation between symmetries at the horizon to that at null infinity.

- Timelike infinity can be used as a link between the symmetries at the horizon and at null infinity. [Chandrasekaran, Flanagan, and Prabhu; ...]

Asymptotic flatness

- We introduce our notion of asymptotic flatness at timelike infinity. It is based on the corresponding notion introduced by Beig and Schmidt at spatial infinity. [Beig and Schmidt, 81-83]

- We introduce a set of "polar coordinates" $\{\tau, \rho, \theta, \varphi\}$ for Minkowski spacetime,

$$\eta_{\mu\nu} x^\mu x^\nu = -\tau^2, \qquad \frac{r}{t} = \frac{\rho}{\sqrt{1+\rho^2}}, \tag{1.1}$$

- In these coordinates flat spacetime metric takes the form

$$ds^2 = -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{1+\rho^2} + \rho^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right) \tag{1.2}$$

Hyperbolic slicing

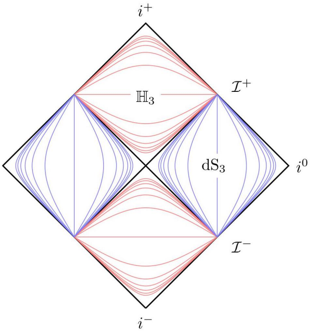

Figure: Hyperbolic slicing of Minkowski space

Asymptotic flatness

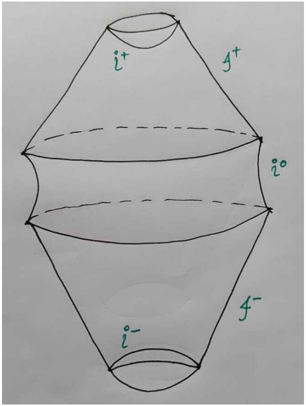

Figure: Zooming in at i^+

Asymptotic flatness: Convenient form

- Consider a general class of spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{m} \ell_{\mu\nu}^{(n)} \frac{1}{\tau^n} + \mathscr{O}(\tau^{-m-1}). \tag{1.3}$$

- Following Beig and Schmidt, this metric can be put in the following more convenient form

$$ds^2 = -N^2 d\tau^2 + h_{ab} d\phi^a d\phi^b, \tag{1.4}$$

where

$$N = 1 + \frac{\sigma(\phi^a)}{\tau}, \tag{1.5}$$

$$h_{ab} = \tau^2 \left[h_{ab}^{(0)}(\phi^c) + \frac{1}{\tau} h_{ab}^{(1)}(\phi^c) + \frac{1}{\tau^2} h_{ab}^{(2)}(\phi^c) + \mathscr{O}\left(\frac{1}{\tau^3}\right) \right]. \tag{1.6}$$

- The above form is our starting point.

Outline

Introduction

Supertranslations at timelike infinity

Charges

Summary

Supertranslations

- If supertranslations are genuine symmetries of general relativity then they should also be realisable at timelike infinity.

- A natural question to ask is what is the set of diffeomorphisms preserving the form of the metric.

Supertranslations

- The following diffeomorphism preserves the asymptotic form of the metric to order $1/\tau$.

$$\tau = \bar{\tau} - \omega(\bar{\phi}^a) + \mathscr{O}\left(\frac{1}{\bar{\tau}}\right), \tag{2.1}$$

$$\phi^a = \bar{\phi}^a + \frac{1}{\bar{\tau}} h^{(0)ab} \partial_b \omega(\bar{\phi}^c) + \mathscr{O}\left(\frac{1}{\bar{\tau}^2}\right), \tag{2.2}$$

- Here $\omega(\phi^a)$ is an arbitrary function on $EAdS_3$ hyperboloid. It determines the higher order terms in the diffeomorphism uniquely.

First order supertranslations

- We only focus on supertranslations in our work.
- Under general supertranslation, the zeroth order field $h_{ab}^{(0)}$ and the first order field σ does not transform,
- The first order metric correction $h_{ab}^{(1)}$ transforms under general supertranslations,

$$h_{ab}^{(1)} \to h_{ab}^{(1)} + 2\mathscr{D}_a\mathscr{D}_b\,\omega - 2\omega h_{ab}^{(0)}. \tag{2.3}$$

Boundary conditions

- ω is an arbitrary function on the hyperboloid. One needs to specify further boundary conditions.
- The boundary conditions at timelike infinity should remain invariant under allowed supertranslations.
- To state the boundary conditions, we define,

$$k_{ab} := h^{(1)}_{ab} + 2\sigma h^{(0)}_{ab}. \tag{2.4}$$

- It follows from that under general supertranslation,

$$k_{ab} \to k_{ab} + 2\mathscr{D}_a\mathscr{D}_b\omega - 2\omega h^{(0)}_{ab}. \tag{2.5}$$

Boundary conditions

- There are two natural sets of boundary conditions to consider.
    - First, one can dispose of supertranslations by demanding that [Cutler; Porrill, 1980s]
    $$k_{ab} = 0.$$
    - Second, taking
    $$k = \operatorname{tr} k_{ab} = 0$$
    while $k_{ab} \neq 0$. [Compere and Dehouck, 2011]
- We work with the second $k = 0$. It implies the following differential equation of ω

$$(\Box - 3)\, \omega = 0. \tag{2.6}$$

Second order supertranslations

- We work with non-linear gravity. Thus, we can obtain second or higher order transformations of the fields.

- The action on the second order field is:

$$h_{ab}^{(2)} \to h_{ab}^{(2)} + \text{terms involving}(\omega k) + \text{terms involving}(\omega\sigma) + \\ + \omega^2 h_{ab}^{(0)} - 2\omega\omega_{ab} + \omega_a^c \omega_{bc} \,. \qquad (2.7)$$

- We find that the second order supertranslations are consistent with the Einstein's equation of motion when split into $3+1$ form.

Outline

Iyer-Wald global charges

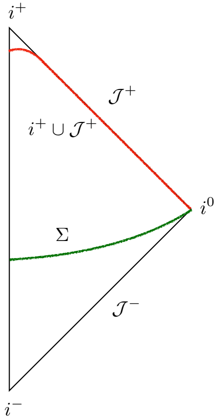

Figure: Spacetime without a horizon

Iyer-Wald charges

- We would like to understand the contributions to the Iyer-Wald global charges from timelike infinity.
- The global charge variation is invariant under local deformations of the Cauchy surface Σ.
- One can deform Σ in the far future to $i^+ \cup \mathscr{I}^+$. In our context

$$\delta Q^{IW}_\xi(\Sigma) = \int_{\mathscr{I}^+} \omega^\gamma n_\gamma \sqrt{h} d^3x + \int_{i^+} \omega^\gamma n_\gamma \sqrt{h} d^3x \tag{3.1}$$

Iyer-Wald charges contd.

- To study the contribution to global charge from i^+, we choose the hypersurface to be a $\tau = \text{constant}$ surface. The volume factor $\sqrt{h}d^3x$ grows as τ^3 in the $\tau \to \infty$ limit.
- On $\tau = \text{constant}$ hypersurface, $-\omega^\gamma n_\gamma = \omega^\tau (1 + \mathscr{O}(1/\tau))$, and the final expression for ω^τ reads,

$$\omega^\tau = \frac{2}{\tau^3}\left(\delta_1 \sigma \delta_2 k - \delta_1 k \delta_2 \sigma\right) + \mathscr{O}(1/\tau^4) \qquad (3.2)$$

- Upon using the boundary condition, $k = 0$, the first variation to Iyer-Wald global charges vanish.

$$\delta Q_{IW}(i^+) = 0. \qquad (3.3)$$

- Since the symplectic flux through timelike infinity is zero, we can define localised charges.

Localised Charges

- Motivated by work at spatial infinity esp., for supertranslations of the form $(\Box - 3)\omega = 0$, we can define a localised charge cf. Compere and Dehouck

$$Q_\omega = -\frac{1}{4\pi G}\int_C \sqrt{q}\, d^2x\, (\sigma\omega_b - \omega\sigma_b) r^b. \tag{3.4}$$

- This is our proposal. A first principal derivation can be given by relating to $\mathscr{I}^+_+$. perhaps following Troessaert and Prabhu

- Here C is spherical cross-section in $\mathscr{H}$. Since there is no symplectic flux, $\Omega(g, \delta_1 g, \delta_2 g) = 0$, this expression is independent of the choice of cross-section C.

- Similarly we can define a localised charge associated with Lorentz symmetries by considering a conserved tensor J_{ab}.

Charges in diagram

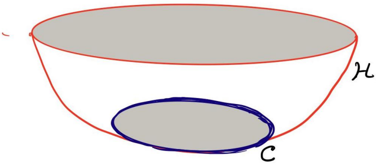

Figure: Spherical crossection surrounding sources in $\mathscr{H}$

Outline

Summary

- We start by defining asymptotic flatness at timelike infinity.
- We study the action of supertranslations at timelike infinity on fields at second order in $1/\tau$ expansion.
- We used our boundary conditions to show that the Lee-Wald symplectic form does not receive contributions from timelike infinity.
- Finally, we proposed expressions for supertranslations and Lorentz charges.

Future directions

- Following the work of Troessaert '17 and Prabhu '19, we expect our charge expressions can be matched with appropriate expressions for supertranslation and Lorentz charges at $\mathscr{I}^+_+$.
- Another important question to answer is: can our boundary conditions be used to relate supertanslations at future null infinity to superstranslations at the horizon?
- The role/action of logarithmic translations, superrotations, etc. at timelike infinity?

THANK YOU

Schwarzschild solution at timelike infinity

- We write the Schwarzschild solution near timelike infinity in the Beig-Schmidt form by doing a series of coordinate transformation.
- The requisite Beig-Schmidt form at first order in expansion of $1/\tau$ not only satisfies $k = 0$ but also satisfies $k_{ab} = 0$.
- The field σ takes the value,

$$\sigma = -(GM)\left(\rho^{-1} + 2\rho\right), \qquad \Box\sigma = 3\sigma. \tag{4.1}$$

Schwarzschild solution at timelike infinity contd.

- We notice that as τ goes to infinity for fixed r, ρ goes to 0. Thus, the horizon $r = 2GM$ intersects the timelike infinity hyperboloid $\mathscr{H}$ at the origin $\rho = 0$.

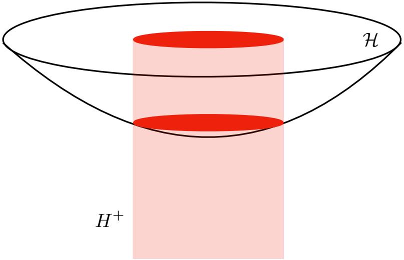

Figure: Timelike infinity

Schwarzschild solution: Sources and Charges

- For the Schwarzschild solution the fields σ and $h^{(1)}_{ab} = -2\sigma h^{(0)}_{ab}$ are singular at $\rho = 0$.

- The charge integral is finite even on a $\rho = \varepsilon$ spherical surface C.

$$Q_\omega = -\frac{1}{4\pi G}\int_C \sqrt{q}\, d^2x\, (\sigma\omega_b - \omega\sigma_b) r^b. \tag{4.2}$$

- For the region $r > 2GM$ of the Schwarzschild solution, timelike infinity is taken to be the hyperboloid $\mathscr{H}$ minus the origin.

- Thus we see that for a gravitationally bound system the origin at timelike inifinity acts as a source for the charge integrals.