On semiclassical singularity theorems

Eleni-Alexandra Kontou

in collaboration with Christopher Fewster, Jackson Fliss, Ben Freivogel and Dimitrios Krommydas

Chennai Symposium on Gravitation and Cosmology
February 5, 2022

UNIVERSITY OF AMSTERDAM

Outline

Based on ArXiv: 2012.11569, 2108.12668 and 2111.05772

Singularity theorems structure

Definition

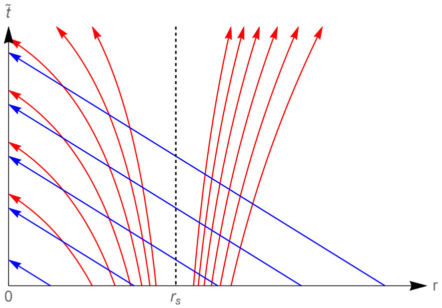
A spacetime is singular if it possesses at least one incomplete geodesic.

Singularity theorems structure

1. **The initial or boundary condition**
   There exists a trapped surface (null geodesics) or a spatial slice with negative expansion (timelike goedesics)

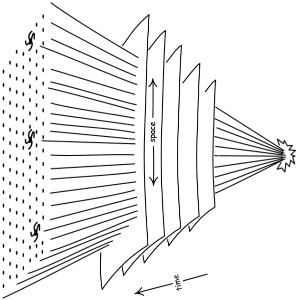
Singularity theorems structure

1. **The initial or boundary condition**
   There exists a trapped surface (null geodesics) or a spatial slice with negative expansion (timelike goedesics)



Singularity theorems structure

1. **The initial or boundary condition**
   There exists a trapped surface (null geodesics) or a spatial slice with negative expansion (timelike goedesics)



Singularity theorems structure

2. **The energy condition**
   Restriction on the stress-energy tensor expressing "physical" properties of matter.

Null geodesics: Null energy condition (NEC) ℓ^{μ}: null vector
Timelike geodesics: Strong energy condition (SEC) U^{μ}: timelike vector

Singularity theorems structure

2. **The energy condition**
   Restriction on the stress-energy tensor expressing “physical” properties of matter.

Null geodesics: Null energy condition (NEC) ℓ^μ: null vector
Timelike geodesics: Strong energy condition (SEC) U^μ: timelike vector

Physical form	Geometric form	Perfect fluid
$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$	$R_{\mu\nu}\ell^\mu\ell^\nu \geq 0$	$\rho + P \geq 0$
$(T_{\mu\nu} - \frac{Tg_{\mu\nu}}{n-2})U^\mu U^\nu \geq 0$	$R_{\mu\nu}U^\mu U^\nu \geq 0$	$\rho + P \geq 0$ and $(n-3)\rho + (n-1)P \geq 0$

Singularity theorems structure

3. **Causality condition**
   There is a Cauchy surface: spacelike hypersurface which intersects causal geodesics once and only once

Singularity theorems structure

3. **Causality condition**
   There is a Cauchy surface: spacelike hypersurface which intersects causal geodesics once and only once

Proof structure

1. Initial condition: Geodesics start focusing
2. Energy condition: Focusing continues
3. Causality condition: No focal points

$\Rightarrow$ Geodesic incompleteness

From classical to semiclassical singularity theorems

Problem

Pointwise energy conditions are violated by many classical and all quantum fields

$$T_{\mu\nu} U^{\mu} U^{\nu} \geq 0$$

From classical to semiclassical singularity theorems

Problem

Pointwise energy conditions are violated by many classical and all quantum fields

$$T_{\mu\nu} U^\mu U^\nu \geq 0$$

Question

Can we have singularity theorems with weaker energy conditions?

$$\int_\gamma f^2 T_{\mu\nu} U^\mu U^\nu \geq -(Bound)$$

From classical to semiclassical singularity theorems

Problem

Pointwise energy conditions are violated by many classical and all quantum fields

$$T_{\mu\nu}U^\mu U^\nu \geq 0$$

Question

Can we have singularity theorems with weaker energy conditions?

$$\int_\gamma f^2 T_{\mu\nu}U^\mu U^\nu \geq -(Bound)$$

Question

Can we have semiclassical singularity theorems?

Singularity theorems with weakened energy conditions

Theorem [Fewster, E-AK, 2019]

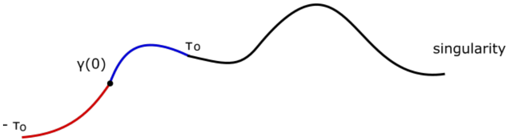
1. Energy condition

$$\int_0^\tau f(t)^2 \overbrace{R_{\mu\nu}U^\mu U^\nu}^{\rho}\, dt \geq -Q_m \|f^{(m)}\|^2 - Q_0\|f\|^2, \quad \|f\|^2 = \int_\gamma f^2 dt$$

and Scenario 1: $\rho \geq 0$ for $[0, \tau_0]$: SEC obeyed for a short time
or Scenario 2: $\rho < 0$ for $[-\tau_0, 0]$: SEC violated before we measure K

2. Initial condition: $K \leq -\nu(Q_m, Q_0, \tau_0, \tau)$
3. Causality condition: There exists a Cauchy surface.

$\Rightarrow$ The spacetime is geodesically incomplete.



Towards semiclassical singularity theorems

(A) Prove quantum energy inequalities (QEIs) for the relevant quantities for timelike and null geodesics
Example of a QEI (bound on energy density in Minkowski spacetime)

$$\int dt\, f^2 \langle :T_{\mu\nu} U^\mu U^\nu: \rangle_\omega \geq -\frac{1}{16\pi^2} \int f''(t)^2 dt$$

[Ford, Roman, 1995], [Fewster, Eveson, 1998]

Towards semiclassical singularity theorems

(A) Prove quantum energy inequalities (QEIs) for the relevant quantities for timelike and null geodesics
Example of a QEI (bound on energy density in Minkowski spacetime)

$$\int dt\, f^2 \langle : T_{\mu\nu} U^\mu U^\nu : \rangle_\omega \geq -\frac{1}{16\pi^2} \int f''(t)^2 dt$$

[Ford, Roman, 1995], [Fewster, Eveson, 1998]

(B) Singularity theorems require a geometric condition. Use of the semiclassical Einstein equation

$$8\pi G_N \langle T_{\mu\nu} \rangle_\omega = G_{\mu\nu} \, .$$

Towards semiclassical singularity theorems

(A) Prove quantum energy inequalities (QEIs) for the relevant quantities for timelike and null geodesics
Example of a QEI (bound on energy density in Minkowski spacetime)

$$\int dt\, f^2 \langle :T_{\mu\nu} U^\mu U^\nu: \rangle_\omega \geq -\frac{1}{16\pi^2} \int f''(t)^2 dt$$

[Ford, Roman, 1995], [Fewster, Eveson, 1998]

(B) Singularity theorems require a geometric condition. Use of the semiclassical Einstein equation

$$8\pi G_N \langle T_{\mu\nu} \rangle_\omega = G_{\mu\nu} \,.$$

(C) Estimate the required initial contraction for physical spacetimes

(A) Quantum strong energy inequality

Difference QEIs

$$\int_\gamma \langle{:}\rho{:}\rangle_\omega f^2 = \int_\gamma \langle\rho\rangle_{\underset{\text{state of interest}}{\overset{\uparrow}{\omega}}} f^2 dt - \int_\gamma \langle\rho\rangle_{\underset{\text{reference state}}{\overset{\uparrow}{\omega_0}}} f^2 dt \geq -\langle \overbrace{\mathfrak{Q}_{\omega_0}(f)}^{\text{bound}} \rangle_\omega \, .$$

(A) Quantum strong energy inequality

Difference QEIs

$$\int_\gamma \langle :\rho: \rangle_\omega f^2 = \int_\gamma \langle \rho \rangle_{\underset{\text{state of interest}}{\underset{\uparrow}{\omega}}} f^2 dt - \int_\gamma \langle \rho \rangle_{\underset{\text{reference state}}{\underset{\uparrow}{\omega_0}}} f^2 dt \geq -\langle \overbrace{\mathfrak{Q}_{\omega_0}(f)}^{\text{bound}} \rangle_\omega \,.$$

For the minimally coupled quantum scalar field:

QSEI

$$\int_\gamma \langle :\rho_U: \rangle_\omega f^2 dt \geq -\int_0^\infty \frac{d\alpha}{\pi} ((\nabla_U \otimes \nabla_U)\, W_0)(\bar{f}_\alpha, f_\alpha) - \frac{M^2}{n-2} \int_\gamma \langle :\phi^2: \rangle_\omega f^2 dt$$

[Fewster, E-AK, 2018]

- W_0:Two-point function of the reference state
- $(\bar{f}_\alpha, f_\alpha)$: Fourier transform
- $\langle :\phi^2: \rangle_\omega$: Wick square of state of interest ω

(B) The semiclassical Einstein equation

Problem

It is generally very difficult (or impossible) to find closed form expressions for two-point functions of reference states in curved spacetimes.

(B) The semiclassical Einstein equation

Problem

It is generally very difficult (or impossible) to find closed form expressions for two-point functions of reference states in curved spacetimes.

Idea

For test functions of sufficiently small support the bound continues to hold if W_0 is the two-point function of Minkowski vacuum.

(B) The semiclassical Einstein equation

Problem

It is generally very difficult (or impossible) to find closed form expressions for two-point functions of reference states in curved spacetimes.

Idea

For test functions of sufficiently small support the bound continues to hold if W_0 is the two-point function of Minkowski vacuum.

For $n = 2m$ spacetime dimensions

$$\int_\gamma \langle :\rho_U: \rangle_\omega f^2 dt \geq -\hbar \frac{\pi S_{2m-2}}{2m(2\pi)^{2m}} \int_\gamma dt |f^{(m)}|^2 - \frac{M^2}{n-2} \int_\gamma \langle :\phi^2: \rangle_\omega f^2 dt$$

(B) The semiclassical Einstein equation

Problem

It is generally very difficult (or impossible) to find closed form expressions for two-point functions of reference states in curved spacetimes.

Idea

For test functions of sufficiently small support the bound continues to hold if W_0 is the two-point function of Minkowski vacuum.

For $n = 2m$ spacetime dimensions

$$\int_\gamma \langle :\rho_U: \rangle_\omega f^2 dt \geq -\hbar \frac{\pi S_{2m-2}}{2m(2\pi)^{2m}} \int_\gamma dt |f^{(m)}|^2 - \frac{M^2}{n-2} \int_\gamma \langle :\phi^2: \rangle_\omega f^2 dt$$

Using the SEE

$$\int d\tau f^2(t) R_{\mu\nu} U^\mu U^\nu \geq -\frac{4\pi\hbar S_{2m-2}}{m(2\pi)^{2m}} ||f^{(m)}||^2 - \frac{4\pi M^2 \phi^2_{\max}}{m-1} ||f||^2 \,,$$

(B) The semiclassical Einstein equation

Problem

It is generally very difficult (or impossible) to find closed form expressions for two-point functions of reference states in curved spacetimes.

Idea

For test functions of sufficiently small support the bound continues to hold if W_0 is the two-point function of Minkowski vacuum.

For $n = 2m$ spacetime dimensions

$$\int_\gamma \langle :\rho_U: \rangle_\omega f^2 dt \geq -\hbar \frac{\pi S_{2m-2}}{2m(2\pi)^{2m}} \int_\gamma dt |f^{(m)}|^2 - \frac{M^2}{n-2} \int_\gamma \langle :\phi^2: \rangle_\omega f^2 dt$$

Using the SEE

$$\int d\tau f^2(t) R_{\mu\nu} U^\mu U^\nu \geq -\frac{4\pi\hbar S_{2m-2}}{m(2\pi)^{2m}} ||f^{(m)}||^2 - \frac{4\pi M^2 \phi^2_{\text{max}}}{m-1} ||f||^2 ,$$

Where we assumed that there is a class of states for which

$$\left| \langle : \phi^2 : \rangle_\omega \right| \leq \phi^2_{\text{max}} ,$$

Partition of unity

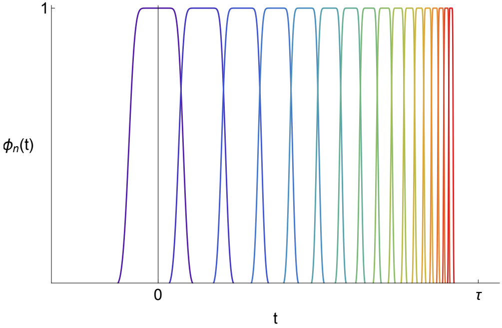
Idea

Break the geodesic into n pieces each of support T_t and sum the resulting inequalities.

Partition of unity

Idea

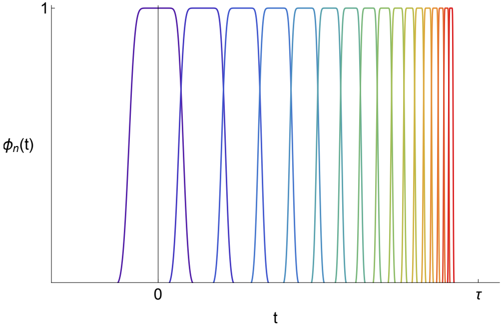
Break the geodesic into n pieces each of support T_t and sum the resulting inequalities.



Partition of unity

Idea

Break the geodesic into n pieces each of support T_t and sum the resulting inequalities.



$$\sum_{n=1}^{\infty} \|(f\phi_n)^{(m)}\|^2 \leq \sum_{j}^{m} c_j(T_0,\tau)\|f^{(j)}\|^2 \equiv |||f|||^2$$

The singularity theorem

[Fewster, E-AK, 2021]

1.

$$\int dt f^2(t) R_{\mu\nu} U^\mu U^\nu \geq -Q_m |||f|||^2 - Q_0 ||f||^2$$

$$Q_m = \frac{\hbar S_{2m-2}}{(2\pi)^{2m-2}}, \qquad \text{and} \qquad Q_0 = \frac{4\pi M^2 \phi_{\max}^2}{m-1}$$

and $R_{\mu\nu} U^\mu U^\nu \geq 0$ holds for $t \in [0, \tau_0]$

3. the initial extrinsic curvature of S satisfies

$$K \leq -\nu(M, \phi_{\max}, T_0, \tau_0, \tau)$$

4. There exists a Cauchy surface.

$\Rightarrow$ The spacetime is timelike geodesically incomplete.

(C) Cosmological application

- We use the ΛCDM model and data from [PLANCK, 2018]: $\Omega_{m0} = 0.31$ and $\Omega_{\Lambda 0} = 0.69$
- The SEC was last satisfied when $t_* = 2.41 \times 10^{17}$s,
$$\boxed{H_* = 3.14 \times 10^{-18}\ \mathrm{s}^{-1}}$$

(C) Cosmological application

- We use the ΛCDM model and data from [PLANCK, 2018]: $\Omega_{m0} = 0.31$ and $\Omega_{\Lambda 0} = 0.69$
- The SEC was last satisfied when $t_* = 2.41 \times 10^{17}$s,
  $H_* = 3.14 \times 10^{-18}\ \mathrm{s}^{-1}$

We want to estimate: $\nu(M, \phi_{\mathrm{max}}, T_0, \tau_0, \tau)$ and compare with H_*

Parameters

M (the mass of the field), ϕ_{max} (the maximum magnitude of the scalar field), T_0 (timescale for valid Minkowski QEI at S), τ (timescale for singularity) and τ_0 (timescale that the SEC is assumed).

(C) Cosmological application

- We use the ΛCDM model and data from [PLANCK, 2018]: $\Omega_{m0} = 0.31$ and $\Omega_{\Lambda 0} = 0.69$
- The SEC was last satisfied when $t_* = 2.41 \times 10^{17}$s, $H_* = 3.14 \times 10^{-18}\ \mathrm{s}^{-1}$

We want to estimate: $\nu(M, \phi_{\max}, T_0, \tau_0, \tau)$ and compare with H_*

Parameters

M (the mass of the field), $\phi_{\max}$ (the maximum magnitude of the scalar field), T_0 (timescale for valid Minkowski QEI at S), τ (timescale for singularity) and τ_0 (timescale that the SEC is assumed).

Particle	optimal τ in s	ν_* in s^{-1}	min T_0 in s
Pion	2.02×10^{20}	3.57×10^{-20}	1.05×10^{-10}
Proton	4.16×10^{18}	1.73×10^{-18}	1.51×10^{11}
Higgs	2.33×10^{14}	3.09×10^{-14}	1.14×10^{-13}

and $\tau_0 \approx T_0/2$

Null quantum energy inequalities

Timelike average of null energy density

For a massless scalar field in Minkowski spacetime

$$\int dt \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\omega f^2(t) \geq -\frac{1}{12\pi^2} \int dt f''(t)^2$$

[Fewster, Roman, 2002]

Null quantum energy inequalities

Timelike average of null energy density

For a massless scalar field in Minkowski spacetime

$$\int dt \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\omega f^2(t) \geq -\frac{1}{12\pi^2} \int dt f''(t)^2$$

[Fewster, Roman, 2002]

Can we have the same QEI over a null geodesic?

$$\int d\lambda \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -c \int d\lambda f''(\lambda)^2$$

Null quantum energy inequalities

Timelike average of null energy density

For a massless scalar field in Minkowski spacetime

$$\int dt \langle :T_{\mu\nu}: \ell^\mu \ell^\nu \rangle_\omega f^2(t) \geq -\frac{1}{12\pi^2} \int dt f''(t)^2$$

[Fewster, Roman, 2002]

Can we have the same QEI over a null geodesic?

$$\int d\lambda \langle :T_{\mu\nu}: \ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -c \int d\lambda f''(\lambda)^2$$

The counterexample

Considered a sequence of vacuum-plus-two-particle states in which the three-momenta of excited modes are unbounded and become more and more parallel to the spatial part of the null vector ℓ^μ. [Fewster, Roman, 2002]

(A) The smeared null energy condition (SNEC)

Idea

In quantum field theory there is often an ultraviolet cutoff ℓ_{UV} which restricts the three-momenta. We can write $G_N \lessapprox \ell_{\mathrm{UV}}^2/N$.

(A) The smeared null energy condition (SNEC)

Idea

In quantum field theory there is often an ultraviolet cutoff ℓ_{UV} which restricts the three-momenta. We can write $G_N \lessapprox \ell_{\mathrm{UV}}^2/N$.

SNEC conjecture

$$\int d\lambda \langle :T_{\mu\nu}:\ell^\mu \ell^\nu\rangle_\omega f^2(\lambda) \geq -\frac{4B}{G_N}\int d\lambda f'(\lambda)^2$$

where B is an undefined number. [Freivogel, Krommydas, 2019]

(A) The smeared null energy condition (SNEC)

Idea

In quantum field theory there is often an ultraviolet cutoff $\ell_{\rm UV}$ which restricts the three-momenta. We can write $G_N \lesssim \ell_{\rm UV}^2/N$.

SNEC conjecture

$$\int d\lambda \langle :T_{\mu\nu}:\ell^\mu \ell^\nu\rangle_\omega f^2(\lambda) \geq -\frac{4B}{G_N}\int d\lambda f'(\lambda)^2$$

where B is an undefined number. [Freivogel, Krommydas, 2019]

It is well-motivated to consider $B \ll 1$. In order to saturate SNEC, we need to saturate the inequality $NG_N \lesssim \ell_{\rm UV}^2$. Not saturated in controlled constructions: the UV cutoff of the theory is far from Planck scale

- Satisfies the Fewster-Roman counterexample ✔
- Proof for free fields in Minkowski [Fliss, Freivogel, 2021] ✔
- Curved spacetimes, interacting fields, limit of $\ell_{\rm UV} \to 0$ ✗

Double null smearing

Idea

Smear over both null directions x_+ and x_-.

For even free massless scalar on Minkowski spacetime

$$\int d^2x^{\pm} g(x^{\pm})^2 \langle T_{--}\rangle_\omega \geq -P_n \left(\int dx^+ (g_+^{(n/2)}(x^+))^2 \right)^{\frac{n-2}{2n}} \times \left(\int dx^- (g_-^{(n/2)}(x^-))^2 \right)^{\frac{n+2}{2n}} .$$

[Fliss, Freivogel, E-AK, 2021]

Double null smearing

Idea

Smear over both null directions x_+ and x_-.

For even free massless scalar on Minkowski spacetime

$$\int d^2x^{\pm} g(x^{\pm})^2 \langle T_{--}\rangle_\omega \geq -P_n \left(\int dx^+ (g_+^{(n/2)}(x^+))^2\right)^{\frac{n-2}{2n}} \times \left(\int dx^- (g_-^{(n/2)}(x^-))^2\right)^{\frac{n+2}{2n}} .$$

[Fliss, Freivogel, E-AK, 2021]

Advantages

- Rigorously proven from a general QEI
- Can be generalized to curved spacetimes
- The smearing can be controlled and does not depend on the theory

(B) SNEC as the energy condition

SNEC

$$\int d\lambda \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -\frac{4B}{G_N} \int d\lambda f'(\lambda)^2$$

(B) SNEC as the energy condition

SNEC

$$\int d\lambda \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -\frac{4B}{G_N} \int d\lambda f'(\lambda)^2$$

Using the SEE, SNEC becomes

$$\int f(\lambda)^2 R_{\mu\nu}\ell^\mu \ell^\nu d\lambda \geq -32\pi B \|f'\|^2 .$$

Same form as

$$\int f(\lambda)^2 R_{\mu\nu}\ell^\mu \ell^\nu \, d\lambda \geq -Q_m \|f^{(m)}\|^2 - Q_0 \|f\|^2,$$

with $m = 1$, $Q_1 = 32\pi B$ and $Q_0 = 0$. [Freivogel, E-AK, Krommydas, 2020]

(C) Application to evaporating black holes

We assume that the metric is well-approximated by Schwarzschild geometry near the horizon so we know the mean normal curvature H of spherically symmetric hypersurfaces.

(C) Application to evaporating black holes

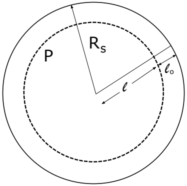
We assume that the metric is well-approximated by Schwarzschild geometry near the horizon so we know the mean normal curvature H of spherically symmetric hypersurfaces.

Plan

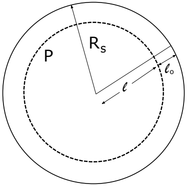
Compare the Schwarzschild mean normal curvature with the required $H < -\nu(Q_1, \tau_0, \tau)$ to satisfy the theorem for the two cases:

- Scenario 1: $\rho \geq 0$ for $[0, \ell_0]$: NEC obeyed for a short time
- Scenario 2: $\rho < 0$ for $[-\ell_0, 0]$: NEC violated before we measure H

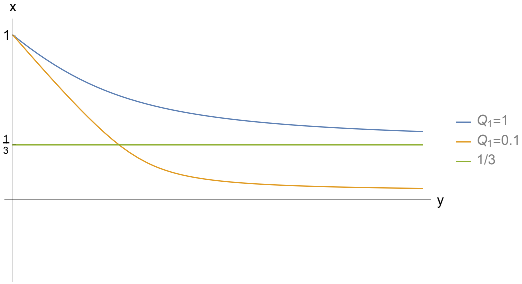
(C) Application to evaporating black holes



(C) Application to evaporating black holes



Comparison with calculated K for Schwarzschild black holes



- $\ell \to yR_s$
- $\ell_0 \to xR_s$

[Freivogel, E-AK, Krommydas, 2020]

Conclusions and future work

Timelike geodesics

- Semiclassical singularity theorem with a QSEI for minimally coupled massive scalar fields
- Curvature approximated using a partition of unity

Conclusions and future work

Timelike geodesics

- Semiclassical singularity theorem with a QSEI for minimally coupled massive scalar fields
- Curvature approximated using a partition of unity
- Future work: Calculation of curvature using a absolute (Hadamard renormalized QEI)

Conclusions and future work

Timelike geodesics

- Semiclassical singularity theorem with a QSEI for minimally coupled massive scalar fields
- Curvature approximated using a partition of unity
- Future work: Calculation of curvature using a absolute (Hadamard renormalized QEI)

Null geodesics

- Motivated the use of a null QEI, the SNEC

Conclusions and future work

Timelike geodesics

- Semiclassical singularity theorem with a QSEI for minimally coupled massive scalar fields
- Curvature approximated using a partition of unity
- Future work: Calculation of curvature using a absolute (Hadamard renormalized QEI)

Null geodesics

- Motivated the use of a null QEI, the SNEC
- This condition can be used to prove a null singularity theorem

Conclusions and future work

Timelike geodesics

- Semiclassical singularity theorem with a QSEI for minimally coupled massive scalar fields
- Curvature approximated using a partition of unity
- Future work: Calculation of curvature using a absolute (Hadamard renormalized QEI)

Null geodesics

- Motivated the use of a null QEI, the SNEC
- This condition can be used to prove a null singularity theorem
- Future work: Use the double smeared null energy condition in a singularity theorem

Conclusions and future work

Timelike geodesics

- Semiclassical singularity theorem with a QSEI for minimally coupled massive scalar fields
- Curvature approximated using a partition of unity
- Future work: Calculation of curvature using a absolute (Hadamard renormalized QEI)

Null geodesics

- Motivated the use of a null QEI, the SNEC
- This condition can be used to prove a null singularity theorem
- Future work: Use the double smeared null energy condition in a singularity theorem

Cosmological and evaporating black hole toy models support the idea that singularities are predicted semi classically but open questions remain.