Hartle-Hawking state in de Sitter spacetime

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De Sitter spacetime

Maximal analytic extension of the spacetime of inflationary universe:

$$ds^2 = -d\tau^2 + \frac{1}{H^2}\cosh^2(H\tau)d\Omega^2\,.$$

$d\Omega^2$: the metric on the unit 3-sphere; H: the Hubble constant.

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$$ds^2 = -d\tau^2 + \cosh^2\tau\, d\chi^2 \,,$$

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χ parametrises a circle of radius 1: $\chi \sim \chi + 2\pi$. We take $-\pi < \chi \leq \pi$.
With $\sinh\tau = \tan T$:

$$ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\chi^2) \,.$$

$-\pi/2 < T < \pi/2$, $-\pi < \chi \leq \pi$.

De Sitter spacetime

2D de Sitter spacetime: the hypersurface

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = 1(= 1/H^2)\,,$$

in 3D Minkowski spacetime with metric

$$ds_M^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2\,.$$

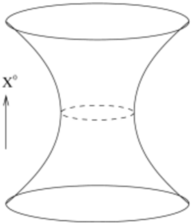

Figure: from Les Houches Lectures in de Sitter Space by Spradlin, Strominger and Volovich

De Sitter spacetime

With

$$X^0 = \tan T\,, \quad X^1 = \frac{\cos\chi}{\cos T} \quad X^2 = \frac{\sin\chi}{\cos T}\,,$$

$$ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\chi^2)\,.$$

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$$ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\chi^2)\,.$$

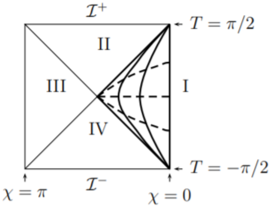
If we let

$$X^0 = \sin\theta\sinh t\,, \quad X^1 = \sin\theta\cosh t\,, \quad X^2 = \cos\theta\,, 0 < \theta < \pi\,.$$

$$\begin{aligned} ds^2 &= -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 \\ &= -\sin^2\theta\, dt^2 + d\theta^2\,. \end{aligned}$$

Static patch: $X^1 > 0$ and $-X^1 < X^0 < X^1$.

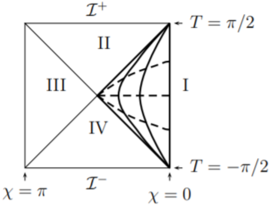
de Sitter spacetime: Carter-Penrose diagram



$$\text{global:}\quad ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\chi^2)\,, -\pi/2 < T < \pi/2.$$

$$\text{static:}\quad ds^2 = -\sin^2\theta\, dt^2 + d\theta^2\,,\quad 0 < \theta < \pi\,.$$

de Sitter spacetime: Carter-Penrose diagram



$$\text{global:} \quad ds^2 = \frac{1}{\cos^2 T}(-dT^2 + d\chi^2)\,, -\pi/2 < T < \pi/2.$$

$$\text{static:} \quad ds^2 = -\sin^2\theta\, dt^2 + d\theta^2\,, \quad 0 < \theta < \pi\,.$$

The static coordinates cover only Region I.
Region I (Right): $(X^0, X^1, X^2) = (\sin\theta \sinh t, \sin\theta\cosh t, \cos\theta)$,
$X_1 > 0$, $-X^1 < X^0 < X^1$.
Region III (Left):
$(X^0, X^1, X^2) = (-\sin\theta\sinh t, -\sin\theta\cosh t, \cos\theta)$,
$X_1 < 0$, $X^1 < X^0 < -X^1$.

Thermal state and Euclidean field theory

Quantum (scalar) field theory in static spacetime:

$$ds^2 = -f(\mathbf{x})dt^2 + g_{ab}(\mathbf{x})dx^a dx^b.$$

A (possibly mixed) state ρ is uniquely determined by the N-point functions

$$\Delta(x_1, x_2, \ldots, x_N) = \mathrm{Tr}\left\{\rho\mathcal{T}\left[\phi(x_1)\phi(x_2)\cdots\phi(x_N)\right]\right\},$$

where $\rho = \sum_{n,m} |m\rangle\rho_{mn}\langle n|$ is a density matrix (i.e. the state) which is Hermitian, positive and satisfies $\mathrm{Tr}\rho = 1$.
$(\mathrm{Tr}\,\Omega = \sum_n \langle n|\Omega|n\rangle)$

- Let $t = -it_{\mathrm{E}}$:

$$ds^2 = f(\mathbf{x})dt_{\mathrm{E}}^2 + g_{ab}(\mathbf{x})dx^a dx^b,$$

  and identify the points $(t_{\mathrm{E}} + \beta, \mathbf{x})$ with $(t_{\mathrm{E}}, \mathbf{x})$.
- The N-point functions can be defined in the Euclidean quantum field theory:
  - The propagator $\Delta(x, y)$ is the unique Green's function satisfying

$$\left[-\nabla_a \nabla^a + m^2\right] \Delta(x, y) = \delta(x, y)\,.$$

  - The N-point correlation function is calculated by the usual Feynman rules with this propagator.
- The state defined by the N-point functions obtained by analytic continuation from those in this Euclidean field theory is a thermal state in the original Lorentzian field theory with temperature T where $\beta = 1/k_B T$.

The Hartle-Hawking state in the static patch of de Sitter spacetime

The metric in the static patch:

$$ds^2 = -\sin^2\theta\, dt^2 + d\theta^2\,, \quad 0 < \theta < \pi\,.$$

Let $t = -it_E$:

$$ds^2 = d\theta^2 + \sin^2\theta\, dt_E^2\,. \quad 0 < \theta < \pi\,.$$

If we identify $t_E \sim t_E + 2\pi$, then this is a smooth sphere.

The Hartle-Hawking state in the static patch of de Sitter spacetime

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If we identify $t_E \sim t_E + 2\pi$, then this is a smooth sphere.
The Hartle-Hawking state is defined as follows:

- The N-point correlation functions are defined in the Euclidean field theory.
- The N-point functions in the static patch of de Sitter space is obtained by analytic continuation.
- These N-point functions define a thermal state with inverse temperature $1/k_B T = 2\pi/H$, i.e. $k_B T = H/2\pi$.

Static patch $\rightarrow$ sphere ($t = -it_E$):

$$X_0 = \sin\theta \sinh t\,, \quad X^1 = \sin\theta \cosh t\,, \quad X^2 = \cos\theta$$
$$\longrightarrow X^0 = -iX^0_{(E)} = -i\sin\theta \sin t_E\,, \quad X^1 = \sin\theta \cos t_E\,, \quad X^2 = \cos\theta\,,$$
$$0 < t_E < 2\pi\,.$$

We have the sphere $(X^0_{(E)})^2 + (X^1)^2 + (X_2)^2 = 1$.

Static patch $\rightarrow$ sphere ($t = -it_E$):

$$X_0 = \sin\theta \sinh t\,, \quad X^1 = \sin\theta\cosh t\,, \quad X^2 = \cos\theta$$

$$\longrightarrow X^0 = -iX^0_{(E)} = -i\sin\theta\sin t_E\,, \quad X^1 = \sin\theta\cos t_E\,, \quad X^2 = \cos\theta\,,$$

$$0 < t_E < 2\pi\,.$$

We have the sphere $(X^0_{(E)})^2 + (X^1)^2 + (X_2)^2 = 1.$

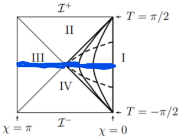
But this sphere can be reached by analytic continuation from the whole spacetime:

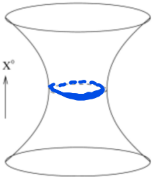
$$X^0 = \tan T\,, \quad X^1 = \frac{\cos\chi}{\cos T}\,, \quad X^2 = \frac{\sin\chi}{\cos T}\,,$$

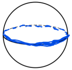
$$(T = -iT_{(E)})$$

$$\longrightarrow X^0 = -iX^0_{(E)} = -i\tanh T_{(E)}\,, X^1 = \frac{\cos\chi}{\cosh T_E}\,, \quad X^2 = \frac{\sin\chi}{\cosh T_{(E)}}\,.$$

In particular the $X^0 = 0$ circle $((X^1)^2 + (X^2)^2 = 1)$ in de Sitter spacetime and the equator $X^0_{(E)} = 0$ $((X^1)^2 + (X^2)^2 = 1)$ of the Euclidean sphere are the same and include both the Right and Left Regions:

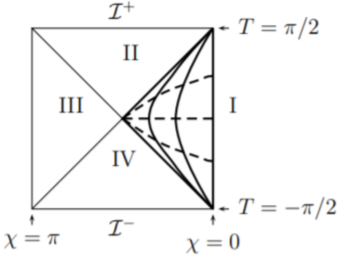






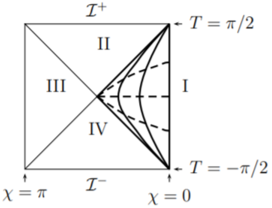
To find the N-point functions on this circle, there is no need for analytic continuation. One can simply calculate them in Euclidean field theory.

Analytic continuation from the Euclidean field theory on the sphere defines N-point functions on the whole de Sitter spacetime. What is this state on the whole de Sitter spacetime?



The state in the whole de Sitter spacetime is a pure state (purified KMS state) with entanglement between the fields in the Regions I and III such that the state restricted to region I is the thermal state described before. Jacobson (by path-integral) 1994; Sewell (by axiomatic field theory) 1982. In this talk I explain this statement in the context of perturbation theory.

Purified KMS state



Wedge-reflection operator J: $(X^0, X^1) \leftrightarrow (-X^0, -X^1)$

Region I: $(X^0, X^1, X^2) = (\sin\theta \sinh t, \sin\theta \cosh t, \cos\theta)$:
field $\phi^{(\mathrm{R})}(t, \theta)$. (Right)

Region III: $(X^0, X^1, X^2) = (-\sin\theta \sinh t, -\sin\theta \cosh t, \cos\theta)$:
field $\phi^{(\mathrm{L})}(t, \theta)$. (Left) [$t$ runs backwards.]

$J\phi^{(\mathrm{R})}(t, \theta)J = \phi^{(\mathrm{L})}(t, \theta)$; $J\phi^{(\mathrm{L})}(t, \theta)J = \phi^{(\mathrm{R})}(t, \theta)$.

The operator J is anti-unitary and $J^2 = \mathbb{I}$.

Purified KMS state

$|n^{(\mathrm{R})}\rangle$: the energy eigenstates on the Right with eigenvalues E_n.

$|n^{(\mathrm{L})}\rangle$: the energy eigenstates on the Left with eigenvalues E_n.

$J|n^{(\mathrm{L})}\rangle = |n^{(\mathrm{R})}\rangle$, $J|n^{(\mathrm{R})}\rangle = |n^{(\mathrm{L})}\rangle$.

H: The Hamiltonian (energy) operator on the Right.

The purified KMS state with inverse temperature β:

$$|\Omega_{\mathrm{KMS}}\rangle = \frac{1}{\sqrt{\mathrm{Tr}(e^{-\beta H})}} \sum_{n^{(\mathrm{R})}} e^{-\beta E_n/2} |n^{(\mathrm{L})}\rangle \otimes |n^{(\mathrm{R})}\rangle \, .$$

This is a pure state.

$$|\Omega_{\rm KMS}\rangle = \frac{1}{\sqrt{{\rm Tr}(e^{-\beta H})}} \sum_{n^{\rm (R)}} e^{-\beta E_n/2} |n^{\rm (L)}\rangle \otimes |n^{\rm (R)}\rangle \,.$$

If the operator $A^{\rm (R)}$ acts on the Right, then

$$\begin{aligned}
&\langle\Omega_{\rm KMS}|A^{\rm (R)}|\Omega_{\rm KMS}\rangle \\
&= \frac{1}{{\rm Tr}(e^{-\beta H})} \sum_{n^{\rm (R)}, n'^{\rm (R)}} e^{-\beta(E_n+E_{n'})/2} \langle n'^{\rm (R)}|A^{\rm (R)}|n^{\rm (R)}\rangle \langle n'^{\rm (L)}|n^{\rm (L)}\rangle \\
&= \frac{1}{{\rm Tr}(e^{-\beta H})} \sum_{n^{\rm (R)}} e^{-\beta E_n} \langle n^{\rm (R)}|A^{\rm (R)}|n^{\rm (R)}\rangle \\
&= \frac{1}{{\rm Tr}(e^{-\beta H})} {\rm Tr}(e^{-\beta H} A^{\rm (R)}) \,.
\end{aligned}$$

$|\Omega_{\rm KMS}\rangle$ gives the thermal state with inverse temperature β on the Right. We'll see that the Hartle-Hawking state is $|\Omega_{\rm KMS}\rangle$.

Purified KMS = Hartle-Hawking

We start from the purified KMS state:

$$|\Omega_{\rm KMS}\rangle = \frac{1}{\sqrt{\mathrm{Tr}(e^{-\beta H})}} \sum_{n^{(\rm R)}} e^{-\beta E_n/2} |n^{(\rm L)}\rangle \otimes |n^{(\rm R)}\rangle \,.$$

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$$\begin{aligned}
&\langle \Omega_{\rm KMS}| \phi^{(\rm L)}(t_1,\theta_1)\phi^{(\rm L)}(t_2,\theta_2)\phi^{(\rm R)}(t_3,\theta_3)\phi^{(\rm R)}(t_4,\theta_4)|\Omega_{\rm KMS}\rangle \\
&= \frac{1}{{\rm Tr}(e^{-\beta H})} \sum_{n'^{(\rm R)},n^{(\rm R)}} e^{-\beta(E_n+E_{n'})/2} \langle n'^{(\rm L)}|\phi^{(\rm L)}(t_1,\theta_1)\phi^{(\rm L)}(t_2,\theta_2)|n^{(\rm L)}\rangle \\
&\quad \times \langle n'^{(\rm R)}|\phi^{(\rm R)}(t_3,\theta_3)\phi^{(\rm R)}(t_4,\theta_4)|n^{(\rm R)}\rangle
\end{aligned}$$

$$\begin{aligned}
&\langle n'^{(\rm L)}|\phi^{(\rm L)}(t_1,\theta_1)\phi^{(\rm L)}(t_2,\theta_2)|n^{(\rm L)}\rangle \\
&= \langle n'^{(\rm L)}|J\phi^{(\rm R)}(t_1,\theta_1)J^2\phi^{(\rm R)}(t_2,\theta_2)J|n^{(\rm L)}\rangle \\
&= \langle Jn'^{(\rm R)}|J\phi^{(\rm R)}(t_1,\theta_1)\phi^{(\rm R)}(t_2,\theta_2)|n^{(\rm R)}\rangle \,,
\end{aligned}$$

where $\langle Jn'^{(\rm R)}|$ is the adjoint of $J|n'^{(\rm R)}\rangle$.

$$
\begin{aligned}
&\langle n'^{(\mathrm{L})}|\phi^{(\mathrm{L})}(t_1,\theta_1)\phi^{(\mathrm{L})}(t_2,\theta_2)|n^{(\mathrm{L})}\rangle \\
&= \langle Jn'^{(\mathrm{R})}|J\phi^{(\mathrm{R})}(t_1,\theta_1)\phi^{(\mathrm{R})}(t_2,\theta_2)|n^{(\mathrm{R})}\rangle\,,
\end{aligned}
$$

$$\langle n'^{(\mathrm{L})}|\phi^{(\mathrm{L})}(t_1,\theta_1)\phi^{(\mathrm{L})}(t_2,\theta_2)|n^{(\mathrm{L})}\rangle$$
$$= \langle Jn'^{(\mathrm{R})}|J\phi^{(\mathrm{R})}(t_1,\theta_1)\phi^{(\mathrm{R})}(t_2,\theta_2)|n^{(\mathrm{R})}\rangle\,,$$

Since J is an anti-unitary operator, $\langle J\psi|J\varphi\rangle = \langle\varphi|\psi\rangle$.

$$\langle n'^{(\mathrm{L})}|\phi^{(\mathrm{L})}(t_1,\theta_1)\phi^{(\mathrm{L})}(t_2,\theta_2)|n^{(\mathrm{L})}\rangle$$
$$= \langle n^{(\mathrm{R})}|\phi^{(\mathrm{R})}(t_2,\theta_2)\phi^{(\mathrm{R})}(t_1,\theta_1)|n'^{(\mathrm{R})}\rangle\,.$$

$$\langle\Omega_{\mathrm{KMS}}|\phi^{(\mathrm{L})}(t_1,\theta_1)\phi^{(\mathrm{L})}(t_2,\theta_2)\phi^{(\mathrm{R})}(t_3,\theta_3)\phi^{(\mathrm{R})}(t_4,\theta_4)|\Omega_{\mathrm{KMS}}\rangle$$
$$= \frac{1}{\mathrm{Tr}(e^{-\beta H})}\sum_{n'^{(\mathrm{R})},n^{(\mathrm{R})}} e^{-\beta(E_n+E_{n'})/2}\langle n'^{(\mathrm{L})}|\phi^{(\mathrm{L})}(t_1,\theta_1)\phi^{(\mathrm{L})}(t_2,\theta_2)|n^{(\mathrm{L})}\rangle$$
$$\times\,\langle n'^{(\mathrm{R})}|\phi^{(\mathrm{R})}(t_3,\theta_3)\phi^{(\mathrm{R})}(t_4,\theta_4)|n^{(\mathrm{R})}\rangle$$

$$\langle\Omega_{\text{KMS}}|\phi^{(\text{L})}(t_1,\theta_1)\phi^{(\text{L})}(t_2,\theta_2)\phi^{(\text{R})}(t_3,\theta_3)\phi^{(\text{R})}(t_4,\theta_4)|\Omega_{\text{KMS}}\rangle$$

$$= \frac{1}{\text{Tr}(e^{-\beta H})}\sum_{n'^{(\text{R})},n^{(\text{R})}} e^{-\beta(E_n+E_{n'})/2}\langle n^{(\text{R})}|\phi^{(\text{R})}(t_2,\theta_2)\phi^{(\text{R})}(t_1,\theta_1)|n'^{(\text{R})}\rangle$$

$$\times\langle n'^{(\text{R})}|\phi^{(\text{R})}(t_3,\theta_3)\phi^{(\text{R})}(t_4,\theta_4)|n^{(\text{R})}\rangle$$

$$= \frac{1}{\text{Tr}(e^{-\beta H})}\sum_{n'^{(\text{R})},n^{(\text{R})}} e^{-\beta E_n/2}\langle n^{(\text{R})}|\phi^{(\text{R})}(t_2,\theta_2)\phi^{(\text{R})}(t_1,\theta_1)|n'^{(\text{R})}\rangle$$

$$\times e^{-\beta E_{n'}/2}\langle n'^{(\text{R})}|\phi^{(\text{R})}(t_3,\theta_3)\phi^{(\text{R})}(t_4,\theta_4)|n^{(\text{R})}\rangle$$

$$= \frac{1}{\text{Tr}(e^{-\beta H})}$$

$$\times\text{Tr}[e^{-\beta H/2}\phi^{(\text{R})}(t_2,\theta_2)\phi^{(\text{R})}(t_1,\theta_1)e^{-\beta H/2}\phi^{(\text{R})}(t_3,\theta_3)\phi^{(\text{R})}(t_4,\theta_4)]$$

$$\langle \Omega_{\rm KMS}|\phi^{(\rm L)}(t_1,\theta_1)\phi^{(\rm L)}(t_2,\theta_2)\phi^{(\rm R)}(t_3,\theta_3)\phi^{(\rm R)}(t_4,\theta_4)|\Omega_{\rm KMS}\rangle$$
$$= \frac{1}{\mathrm{Tr}(e^{-\beta H})}$$
$$\times\ \mathrm{Tr}[e^{-\beta H/2}\phi^{(\rm R)}(t_2,\theta_2)\phi^{(\rm R)}(t_1,\theta_1)e^{-\beta H/2}\phi^{(\rm R)}(t_3,\theta_3)\phi^{(\rm R)}(t_4,\theta_4)]$$

$$e^{iaH}\phi^{(\rm R)}(t,\theta)e^{-iaH} = \phi^{(\rm R)}(t+a,\phi)$$
$$\Rightarrow e^{\beta H/2}\phi^{(\rm R)}(t_1,\phi_1)e^{-\beta H/2} = \phi^{(\rm R)}(t_1 - i\beta/2,\theta)$$
$$\Rightarrow \phi^{(\rm R)}(t_1,\phi_1)e^{-\beta H/2} = e^{-\beta H/2}\phi^{(\rm R)}(t_1 - i\beta/2,\theta).$$

$$\langle \Omega_{\rm KMS}|\phi^{(\rm L)}(t_1,\theta_1)\phi^{(\rm L)}(t_2,\theta_2)\phi^{(\rm R)}(t_3,\theta_3)\phi^{(\rm R)}(t_4,\theta_4)|\Omega_{\rm KMS}\rangle$$
$$= \frac{1}{\mathrm{Tr}(e^{-\beta H})}$$
$$\times\ \mathrm{Tr}[e^{-\beta H}\phi^{(\rm R)}(t_2 - i\beta/2,\theta_2)\phi^{(\rm R)}(t_1 - i\beta/2,\theta_1)$$
$$\times\ \phi^{(\rm R)}(t_3,\theta_3)\phi^{(\rm R)}(t_4,\theta_4)]$$

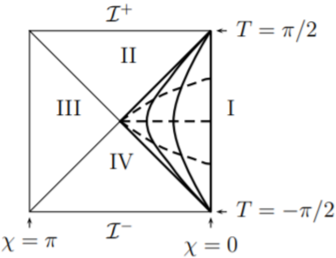
$$\begin{aligned}
&\langle\Omega_{\mathrm{KMS}}|\phi^{(\mathrm{L})}(t_1,\theta_1)\phi^{(\mathrm{L})}(t_2,\theta_2)\phi^{(\mathrm{R})}(t_3,\theta_3)\phi^{(\mathrm{R})}(t_4,\theta_4)|\Omega_{\mathrm{KMS}}\rangle \\
&= \frac{1}{\mathrm{Tr}(e^{-\beta H})} \\
&\quad \times \mathrm{Tr}[e^{-\beta H}\phi^{(\mathrm{R})}(t_2 - i\beta/2,\theta_2)\phi^{(\mathrm{R})}(t_1 - i\beta/2,\theta_1) \\
&\qquad\qquad \times \phi^{(\mathrm{R})}(t_3,\theta_3)\phi^{(\mathrm{R})}(t_4,\theta_4)] \\
&= \frac{1}{\mathrm{Tr}(e^{-\beta H})} \\
&\quad \times \mathrm{Tr}[e^{-\beta H}\phi^{(\mathrm{R})}(t_2 - i\pi,\theta_2)\phi^{(\mathrm{R})}(t_1 - i\pi,\theta_1) \\
&\qquad\qquad \times \phi^{(\mathrm{R})}(t_3,\theta_3)\phi^{(\mathrm{R})}(t_4,\theta_4)]
\end{aligned}$$

because $\beta = 2\pi$.

Purified KMS = Hartle-Hawking

For simplicity let $t_1 = t_2 = t_3 = t_4 = 0$ (all points are on the "equator" and all operators commute). Then, all points are on the Euclidean sphere as well, so there is no need for analytic continuation.

$$\begin{aligned}&\langle\Omega_{\text{KMS}}|\phi^{(\text{L})}(0,\theta_1)\phi^{(\text{L})}(0,\theta_2)\phi^{(\text{R})}(0,\theta_3)\phi^{(\text{R})}(0,\theta_4)|\Omega_{\text{KMS}}\rangle\\ &= \frac{1}{\text{Tr}(e^{-\beta H})}\\ &\qquad \times \text{Tr}[e^{-\beta H}\phi^{(\text{R})}(-i\pi,\theta_1)\phi^{(\text{R})}(-i\pi,\theta_2)\phi^{(\text{R})}(0,\theta_3)\phi^{(\text{R})}(0,\theta_4)]\end{aligned}$$



Purified KMS = Hartle-Hawking

$$
\begin{aligned}
&\langle\Omega_{\rm KMS}|\phi^{(\rm L)}(0,\theta_1)\phi^{(\rm L)}(0,\theta_2)\phi^{(\rm R)}(0,\theta_3)\phi^{(\rm R)}(0,\theta_4)|\Omega_{\rm KMS}\rangle \\
&= \frac{1}{{\rm Tr}(e^{-\beta H})} \\
&\quad \times {\rm Tr}[e^{-\beta H}\phi^{(\rm R)}(-i\pi,\theta_1)\phi^{(\rm R)}(-i\pi,\theta_2)\phi^{(\rm R)}(0,\theta_3)\phi^{(\rm R)}(0,\theta_4)]
\end{aligned}
$$

Right: $(X^1, X^2) = (\sin\theta\cosh t, \cos\theta)$.
Left: $(X^1, X^2) = (-\sin\theta\cosh t, \cos\theta)$.

- The left-hand side is the 4-point function in $|\Omega_{\rm KMS}\rangle$ with the points at $(X^1, X^2) = (-\sin\theta_1, \cos\theta_1)$, $(-\sin\theta_2, \cos\theta_2)$, $(\sin\theta_3, \cos\theta_3)$, $(\sin\theta_4, \cos\theta_4)$;
- The right-hand side is the 4-point function computed in Euclidean field theory with the points at $(X^1, X^2) = (\sin\theta_1\cosh(-i\pi), \cos\theta_1)$, $(\sin\theta_2\cosh(-i\pi), \cos\theta_2)$, $(\sin\theta_3, \cos\theta_3)$, $(\sin\theta_4, \cos\theta_4)$.

Since $\cosh(-i\pi) = \cos(-\pi) = -1$, the points are the same. Thus, $|\Omega_{\rm KMS}\rangle$ is the Hartle-Hawking state.

Summary

- One obtains a sphere by letting the time be imaginary for de Sitter spacetime.
- The Hartle-Hawking state in de Sitter spacetime for interacting scalar field theory is a state with the N-point function obtained by analytically continuing those in the Euclidean field theory on the sphere.
- The Hartle-Hawking state is the purified KMS state, which is a pure state with entanglement between the two static patches and which gives the thermal state with temperature $H/2\pi$ in the static patches.