Surprising consequences of tiny positive cosmological constant in Bondi-Sachs formalism

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EPJC, 81, 696(2021) with Piotr T. Chruściel, Tomasz Smolka, Maciej Maliborski

A brief history

- Well-defined notion of gravitational wave in full non-linear theory of general relativity for asymptotically flat space-time.

- Bondi and his collaborators provided a detailed analysis of gravitational waves in full non-linear theory along with a definition of energy carried away from an isolated system by gravitational radiation.

H. Bondi, M.G.J. Vander Burg, A. W. K. Metzner - 1962

- One of the remarkable milestones in gravitational radiation theory!

- Bondi's analysis has a lot of application in mathematical relativity, numerical relativity, asymptotic symmetries, recently on cosmology!!

A brief history

- Bondi and his collaborators performed a systematic expansions of axis symmetric gravitational wave metric along outgoing null directions.

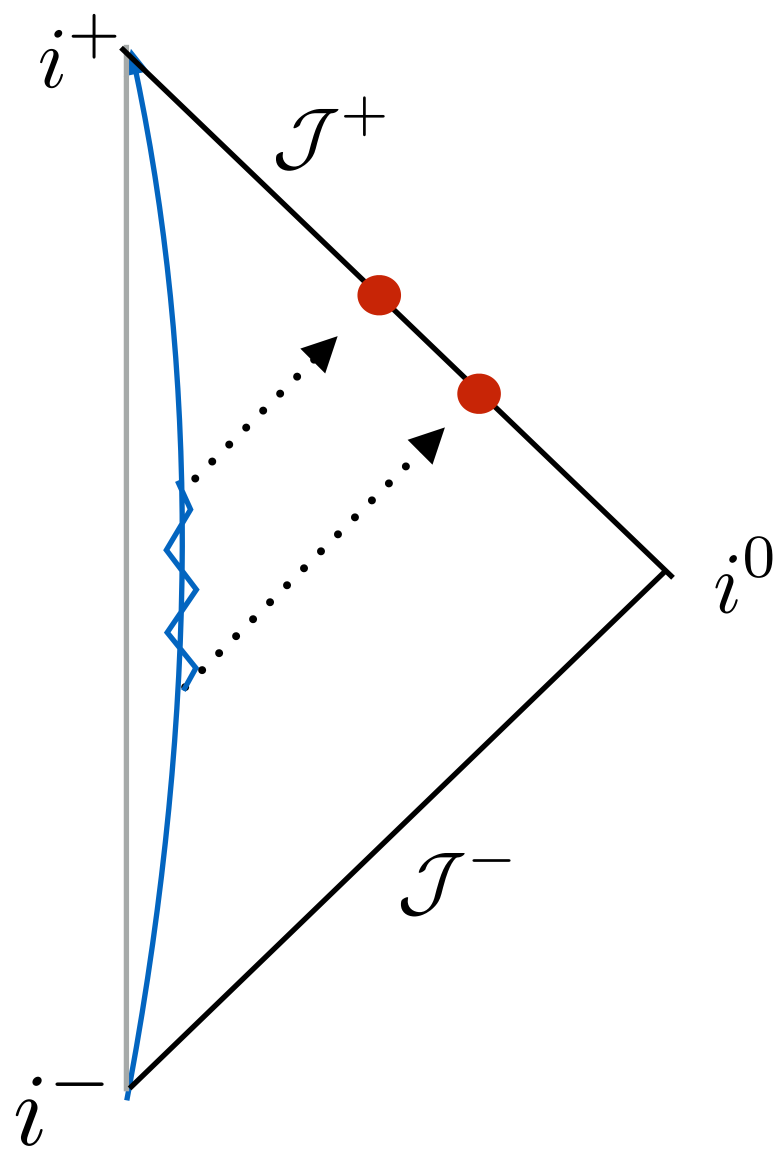
- Given an initial data on null hypersuface, solve Einstein's equations.

- Deduce the asymptotic fall-off condition for the gravitational fields.

- As a supplementary condition obtain mass-loss formula.

- 'News function' - absolute square integrated over the sphere at infinity, measures the rate of energy loss by an isolated system.

$$\frac{dM}{du} = -\int |\partial_u \overset{(-1)}{\check{h}}_{AB}|^2 \sin\theta d\theta d\phi$$



Questions...

- Given that cosmological observations suggest a positive Λ,

  how to generalise Bondi-Sachs's formalism?

  Asymptotic fall-off condition for the gravitational fields?

  Asymptotic symmetries?

  Mass-loss formula for $\Lambda > 0$?

  Analogous 'News tensor'?

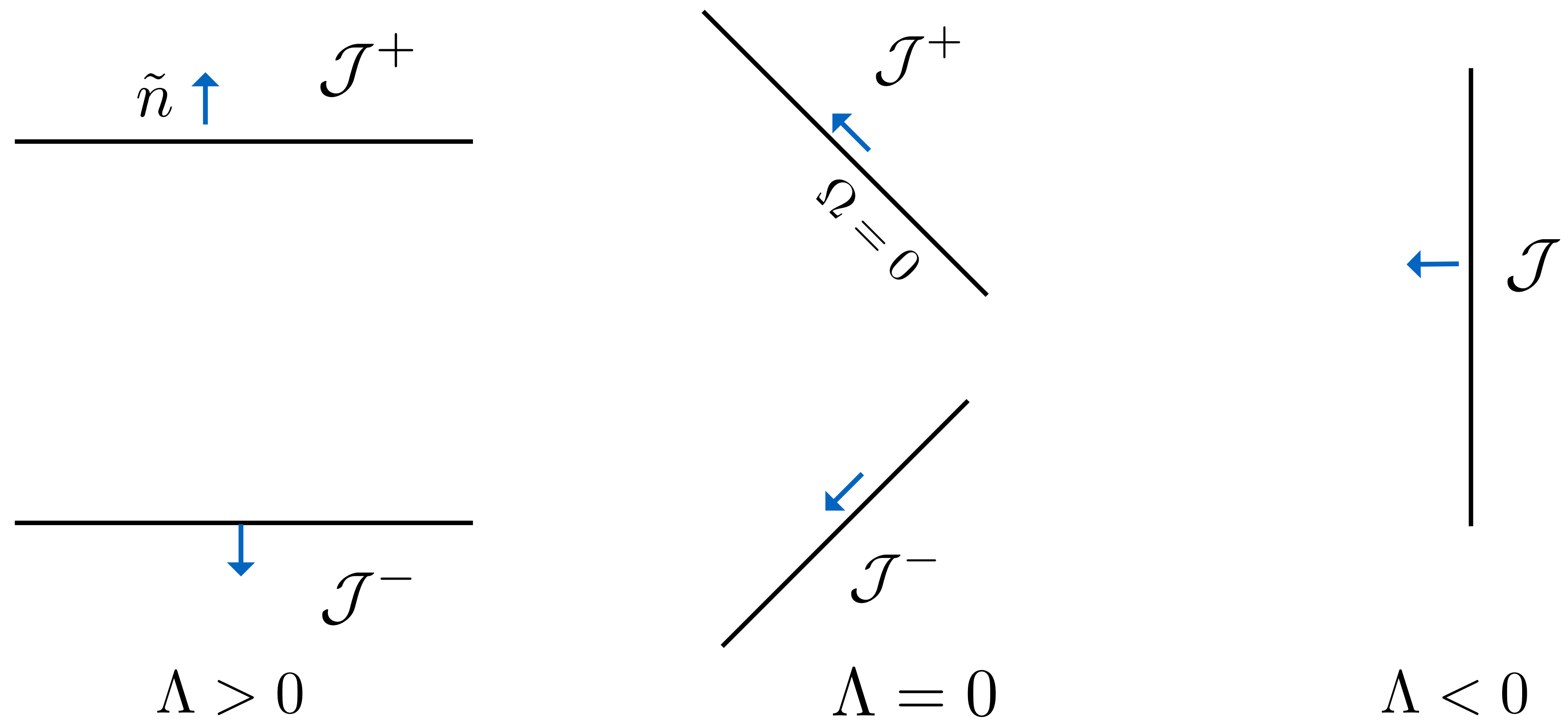
Surprisingly this remained unsolved for 60 years!! Lots of Non-trivialities !!

- A simpler version of the problem - Bondi-Sach's formalism for linearised gravitational fields on de Sitter background.

Non-triviality for positive Λ

- Standard framework does not extend from $\Lambda = 0$ to $\Lambda > 0$.

- Structure of null infinity alters, for $\Lambda > 0$: space-like
  $\Lambda < 0$: time-like



Set up for Bondi coordinates

- We construct Bondi coordinates for de Sitter.

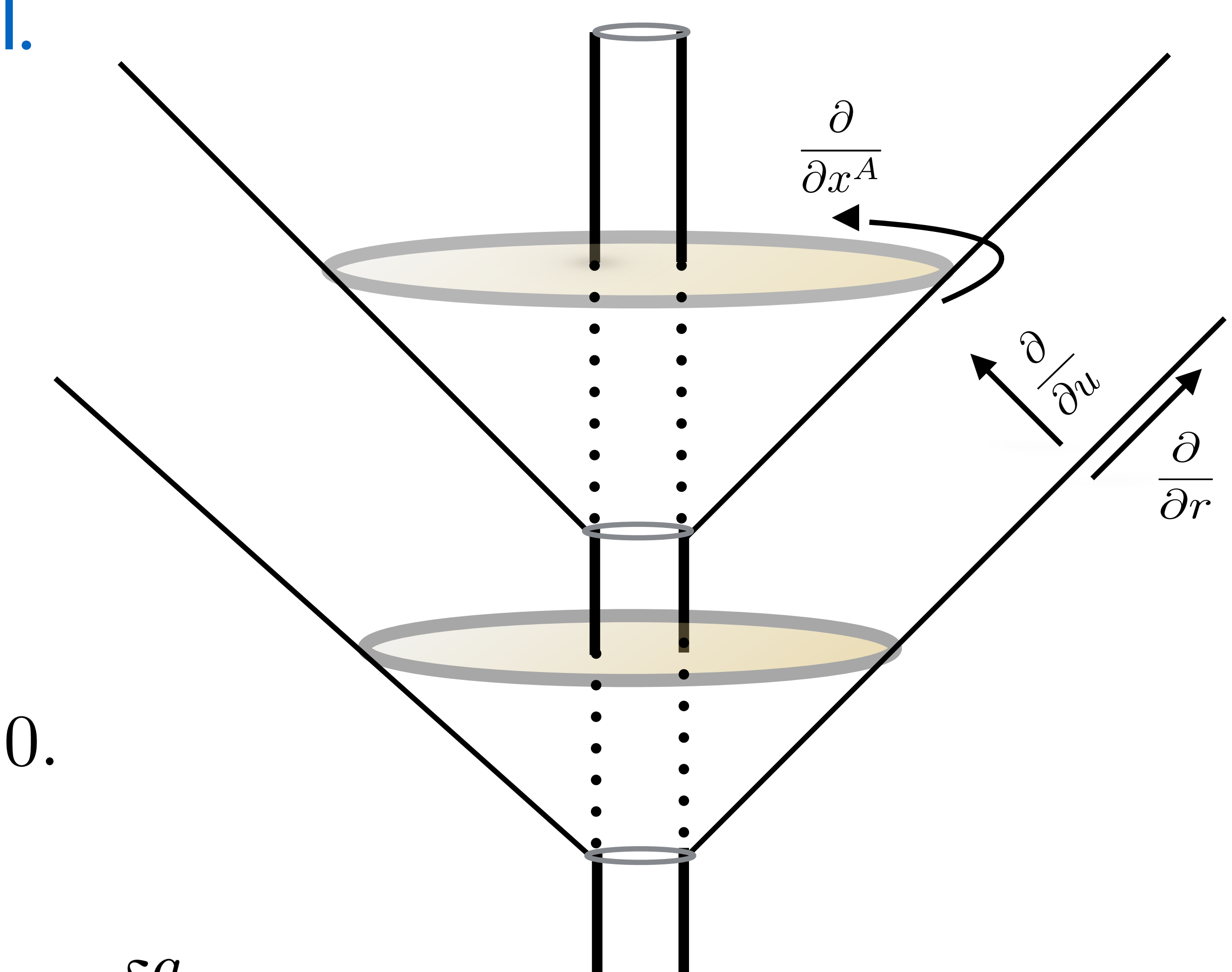
- Bondi coordinates are based on a family of outgoing null hypersurfaces.

- Hypersurfaces $u = const$ are null.

$$\Longrightarrow \quad g^{ab}\partial_a u \partial_b u = 0 \quad \Longrightarrow \quad g^{uu} = 0$$

- Two angular coordinates x^A, are constant along null rays.

$$\Longrightarrow \quad g^{ab}\partial_a u \partial_b x^A = 0 \quad \Longrightarrow \quad g^{uA} = 0.$$



- g^{ab} and g_{ab} are related by $g^{ac}g_{cb} = \delta^a_b$

$$\Longrightarrow \quad g_{rr} = 0 = g_{rA}$$

Metric in Bondi-Sachs coordinates

- Metric in Bondi-Sachs coordinates,

$$ds^2 = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

- r varies along null rays, chosen to be an areal coordinate;

$$\det g_{AB} = r^4 \sin^2\theta$$

- We will explore Einstein equations for linearized fields on de Sitter background.

  background $g_{ab}(\lambda = 0) := \bar{g}_{ab}$; perturbation $h_{ab} := \left.\frac{dg_{ab}(\lambda)}{d\lambda}\right|_{\lambda=0}$

Bondi gauge for linearized theory

- In Bondi coordinates de SitterBackground metric takes the form,

$$\bar{ds}^2 = -\left(1 - \frac{\Lambda r^2}{3}\right) du^2 - 2du dr + r^2 \mathring{\gamma}_{AB} dx^A dx^B$$

- Bondi gauge condition for linearized fields

$$h_{rr} = 0 = h_{rA}, \qquad \mathring{\gamma}^{AB} h_{AB} = 0$$

- We wish to explore linearised Einstein equation with Bondi metric,

$$E_{ab} := R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0$$

Einstein equations: systems of hierarchical PDEs

- Four independent hyper surface equations, $E^u_a = 0$

- $E^u_r = 0 \implies \partial_r \beta = \frac{r}{16}\gamma^{AC}\gamma^{BD}(\partial_r\gamma_{AB})(\partial_r\gamma_{CD})$

  Linearisation: $\partial_r \delta\beta = 0 \implies \delta\beta = \delta\beta(u, x^A)$

Using gauge $\delta\beta = 0 = \delta g_{ur}$, $h_{ab} \mapsto h_{ab} + \mathcal{L}_\xi \bar{g}_{ab}$

- $E^u_A = 0 \implies \partial_r[r^4 e^{-2\beta}\gamma_{AB}(\partial_r U^B)] = 2r^4\partial_r\left(\frac{1}{r}D_A\beta\right) - r^2\gamma^{EF}D_E(\partial_r\gamma_{AF})$

  Linearisation: $\partial_r[r^4\partial_r(r^{-2}\delta g_{uA})] = r^2\mathring{D}^F\partial_r(r^{-2}\delta g_{AF})$

- $$\begin{aligned} E^u_u = 0 \implies 2e^{-2\beta}(\partial_r V) &= \mathcal{R} - 2\gamma^{AB}[D_A D_B\beta + D_A\beta D_B\beta] \\ &+ \frac{e^{-2\beta}}{r^2}D_A[\partial_r(r^4U^A)] - \frac{r^4}{2}e^{-2\beta}\gamma_{AB}(\partial_r U^A)(\partial_r U^B) - 2\Lambda r^2 \end{aligned}$$

  Linearisation: $2\partial_r\delta V = \delta\mathcal{R} - \frac{1}{r^2}\mathring{D}^A[\partial_r(r^2\delta g_{uA})]$

Solution for h_{uA}

Given the ansatz : $$h_{AB} = r^2\left(\overset{(0)}{\check{h}}_{AB} + \frac{\overset{(-1)}{\check{h}}_{AB}}{r} + \frac{\overset{(-2)}{\check{h}}_{AB}}{r^2} + \frac{\overset{(-3)}{\check{h}}_{AB}}{r^3} + \dots\right)$$

solve $$\partial_r[r^4\partial_r(r^{-2}\delta g_{uA})] = r^2\mathring{D}^F\partial_r(r^{-2}\delta g_{AF})$$?

$$h_{uA} = r^2\left(\overset{(0)}{\check{h}}_{uA} + \frac{1}{2}\mathring{D}^B\overset{(-1)}{\check{h}}_{AB}\, r^{-2} + \left(\overset{(-3)}{\check{h}}_{uA} + \frac{2}{9}\mathring{D}^B\overset{(-2)}{\check{h}}_{AB}(3\ln r + 1)\right)r^{-3} + \dots\right)$$

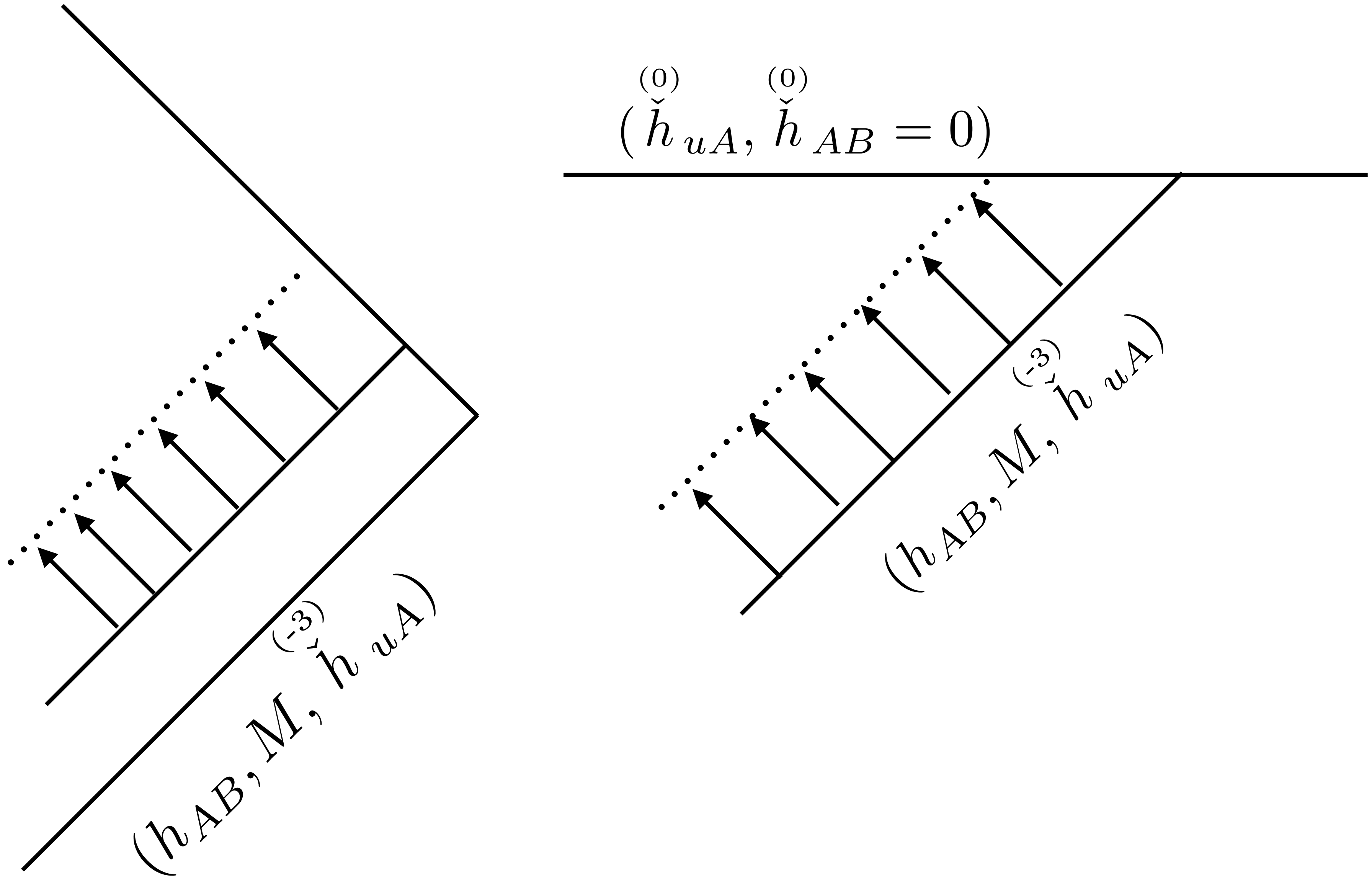
- Similarly, solve for V

- Given the ansatz h_{AB}, hypersuface equations $E^u_a = 0$ fix the asymptotic fall off condition for other components of field.

Evolution equation for $\check{h}_{AB}$

- Traceless symmetric parts of $E_{AB} = 0$ gives evolution equation for $\check{h}_{AB} := r^{-2} h_{AB}$

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2 \check{h}_{AB}))] = 0$$



Non-polyhomogenous de Sitter

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2\check{h}_{AB}))] = 0$$

Asymptotic analysis of this equation gives non trivial equations

$$\frac{\Lambda}{3}\overset{(-2)}{\check{h}}_{AB} = 0$$

- NO log term in de Sitter. De Sitter is non-polyhomogenous!!
- To get rid of log term one needs to set $\overset{(-2)}{\check{h}}_{AB} = 0,$ for flat space-time. In Bondi's paper this condition is termed as outgoing radiation condition.
- For de Sitter this is a consequence of equation of motion.
- This result is true for full non-linear theory also.

G. Compère, A. Fiorucci, R Ruzziconi - 2019

A. Pole, K. Skenderis, M. Taylor -2019

Asymptotic symmetry group is NOT BMS

$$r\partial_r[r(\partial_u\check{h}_{AB})]+\frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2-1)(\partial_r\check{h}_{AB})]-TS[\mathring{D}_A(\partial_r(r^2\check{h}_{AB}))]=0$$

Asymptotic analysis of this equation also gives,

$$\partial_u \overset{(0)}{\check{h}}_{AB}=\frac{\Lambda}{3}\overset{(-1)}{\check{h}}_{AB}+(\mathring{D}_A\overset{(0)}{\check{h}}_{uB}+\mathring{D}_B\overset{(0)}{\check{h}}_{uA}-\mathring{\gamma}_{AB}\mathring{D}^C\overset{(0)}{\check{h}}_{uC})$$

- $\overset{(0)}{\check{h}}_{AB}$ and $\overset{(0)}{\check{h}}_{uA}$ can not be zero simultaneously by a gauge transformation!! Asymptotic symmetry group of de Sitter is not BMS.

- Whether this gauge condition is achieved by any physical space-time is difficult.

S. J. Hoque, A. Virmani - 2021

Asymptotic expansion of linearised fields

$$h_{AB} = r^2\Big(\underbrace{\overset{(0)}{\check{h}}_{AB}}_{=0} + \frac{\overset{(-1)}{\check{h}}_{AB}}{r} + \underbrace{\overset{(-2)}{\check{h}}_{AB}}_{=0} r^{-2} + \frac{\overset{(-3)}{\check{h}}_{AB}}{r^3} + \ldots\Big),$$

$$h_{uA} = r^2\Big(\overset{(0)}{\check{h}}_{uA} + \frac{1}{2}\mathring{D}^B \overset{(-1)}{\check{h}}_{AB} r^{-2} + \overset{(-3)}{\check{h}}_{uA} r^{-3} + \ldots\Big),$$

$$h_{uu} = r\mathring{D}^A \overset{(0)}{\check{h}}_{uA} + \frac{M}{r} - \frac{1}{2r^2}\mathring{D}^A \overset{(-3)}{\check{h}}_{uA} + \ldots$$

$$h_{ur} = 0$$

Evolution equations for integration constant

- $E_{uu} = 0$, gives the evolution equation for h_{uu}

$$2\partial_u M = \partial_u \mathring{D}^A \mathring{D}^B \overset{(-1)}{\check{h}}_{AB} - \Lambda \mathring{D}^A \overset{(-3)}{\check{h}}_{uA}$$

- A non-trivial contribution to memory for cosmological constant!!

- $E_{uA} = 0$, gives the evolution equation for h_{uA}

$$3\partial_u \overset{(-3)}{\check{h}}_{uA} = \mathring{D}_A M + \frac{1}{2}(\mathring{D}^B \mathring{D}_A \mathring{D}^C \overset{(-1)}{\check{h}}_{CB} - \triangle_{\mathring{\gamma}} \mathring{D}^C \overset{(-1)}{\check{h}}_{CA}) - \Lambda \mathring{D}^B \overset{(-3)}{\check{h}}_{AB}$$

Summary

- Bondi-Sachs coordinates are constructed for de Sitter.
- NO log term in de Sitter
- Asymptotic fall off condition for linearised gravitational field have been obtained in Bondi frame. Qualitatively different from $\Lambda = 0$ case.
- Due to different fall-off asymptotic symmetry group is not BMS
- A new contribution is expected in memory effect due to cosmological constant.
- Interesting to generalise Bondi-Sachs formalism for FLRW case.

B. Bonga, K. Prabhu - 2020

Thank you

The residual gauge transformations are thus defined by a u-parameterised family of vector fields $\xi^A(u,\cdot)$ on S^2 together with

$$\partial_u \xi^u(u,x^A) = \frac{\mathring{D}_B \xi^B(u,x^A)}{2}\,, \tag{3.44}$$

and (3.25). Explicitly:

$$\begin{aligned}\mathring{\zeta} = & \left(\int \frac{\mathring{D}_B \xi^B(u,x^A)}{2} du + \mathring{\xi}^u(x^A)\right)\partial_u + \frac{1}{2}\left(\Delta_{\mathring{\gamma}}\xi^u - r\mathring{D}_B\xi^B\right)\partial_r \\ & +(\xi^B(u,x^A) - \frac{1}{r}\mathring{D}^B\xi^u(u,x^A))\partial_B\,,\end{aligned} \tag{3.45}$$

with an arbitrary function $\mathring{\xi}^u(x^A)$.