Spacetime Entanglement Entropy of Quantum Fields

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Work done with Sumati Surya and Nomaan X

Based on Class. Quant. Grav. 39 035004, arXiv:2109.05845

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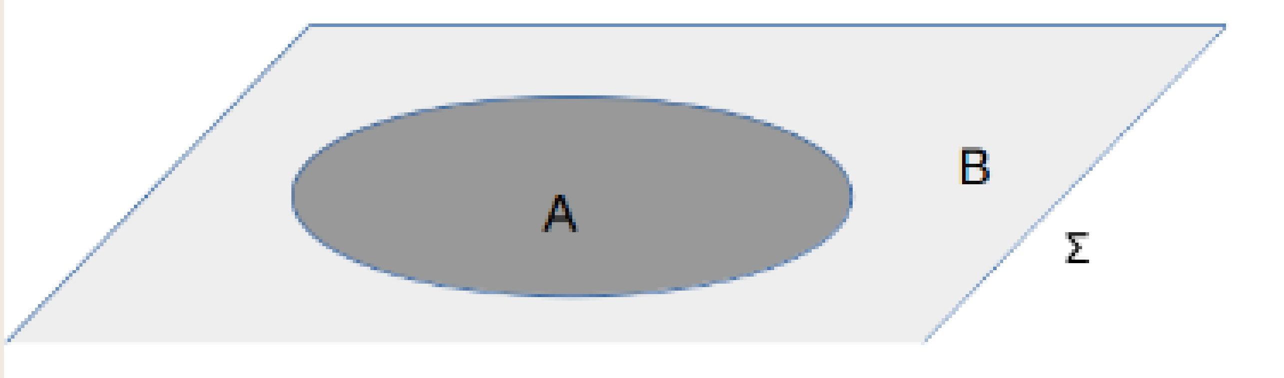
Entanglement entropy of Quantum Fields across black hole horizon as a source of BH entropy

Ref:- L. Bombelli, R. Koul, J. Lee, R. Sorkin, Phys. Rev. D.34. 373

How to measure entanglement of Quantum Fields?

Possible answer: Extend von Neumann entropy to Quantum Field system



$$\rho_A = \mathrm{Tr}_B(\rho) = \sum_{\{|\psi_b\rangle\} \subset \mathcal{F}_B} \langle \psi_b | \rho | \psi_b \rangle$$

$$\mathcal{S}_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

Density matrix is defined at a moment of time (Cauchy hypersurface)

Quantum Gravity theories may not admit Cauchy hypersurface. (Eg: AdS, Topology change, Causal Set Theory etc)

Need for a spacetime definition of QFT entanglement entropy, which will also ensure covariance

Spacetime formula

For a pure state W in $\mathcal{M}$, the SSEE in $\mathcal{O} \subset \mathcal{M}$

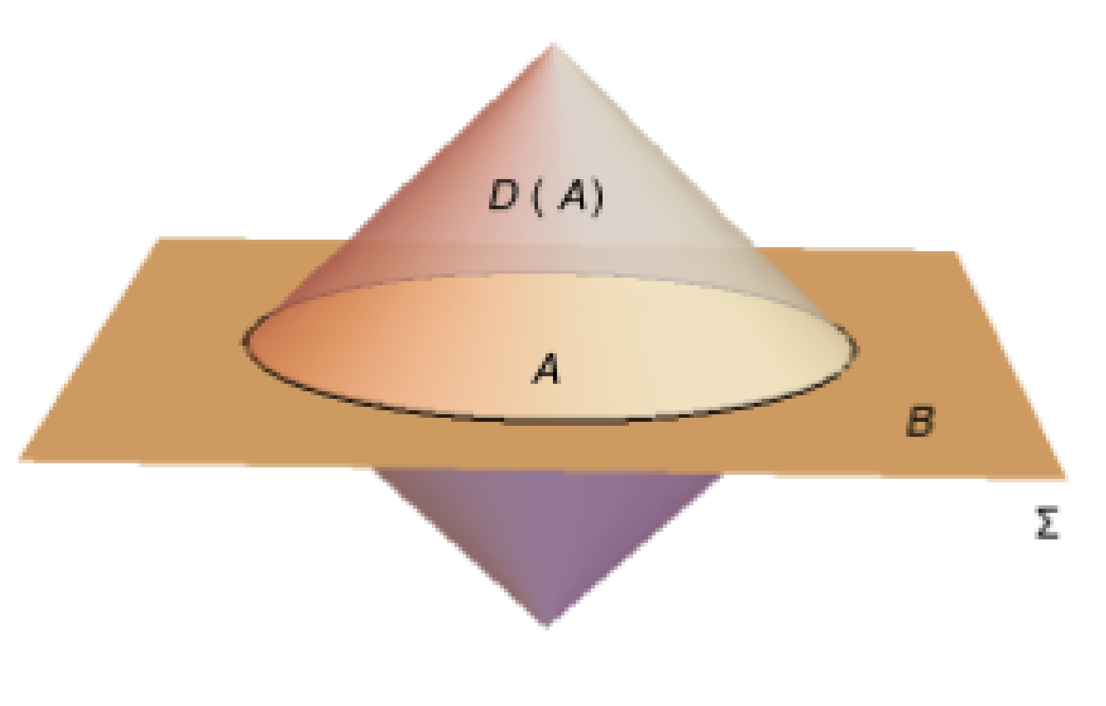
$$W\big|_{\mathcal{O}} \circ \chi = \mu (i\Delta \circ \chi)_{\mathcal{O}}, \quad i\Delta \circ \chi \neq 0, \quad \mathcal{S}_{\mathcal{O}} = \sum_{\mu} \mu \ln |\mu|$$

–R. D. Sorkin, J.Phys.Conf.Ser. 484 (2014) 012004

$$(A \circ f)(x) = \int dV_{x'} A(x;x') f(x')$$

$$i\Delta(x;x') = [\hat{\Phi}(x), \hat{\Phi}(x')]$$

$$W(x;x') = \langle \hat{\Phi}(x) \hat{\Phi}(x') \rangle_{\text{vac}}$$



Well defined if $\mathcal{O}$ is compact

Properties of the generalised eigenvalue (μ)

1) μ's are real

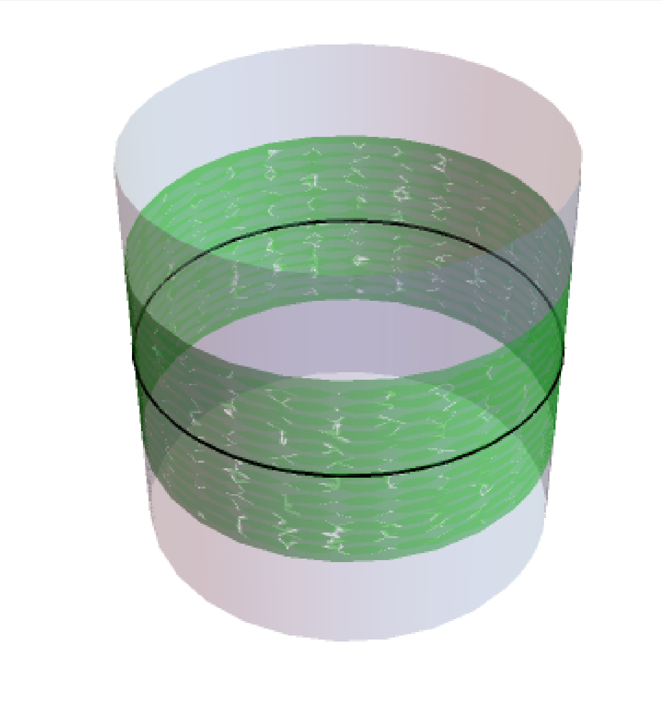
2) Come in pairs (μ,1-μ)

3) $\mu \in (-\infty, 0] \cup [1, \infty)$

$$S_{\mathcal{O}} = \sum_{\mu \geq 1} \left(\mu \ln \mu - (\mu - 1) \ln(\mu - 1) \right) \geq 0$$

Equality holds iff all μ=1

All μ=1 if $W|_{\mathcal{O}}$ is pure



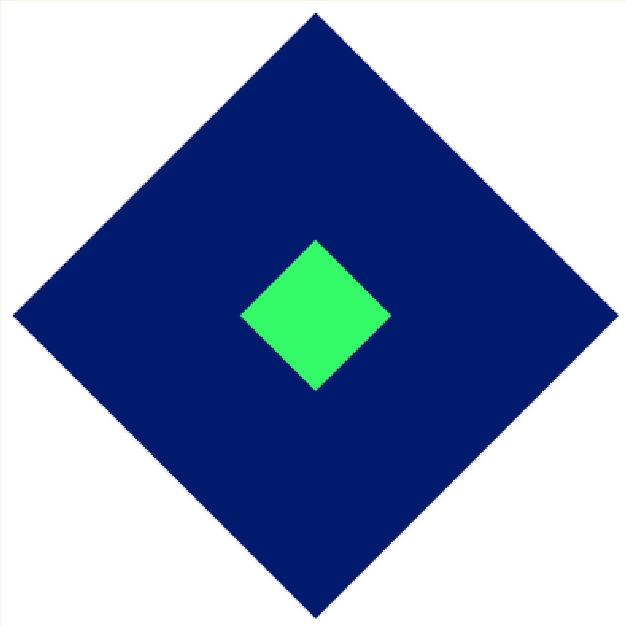
Reproducing the Calabrese-Cardy formula in 2d

P. Calabrese and J. Cardy, J.Stat.Mech.0406:P06002 (2004)

1) Nested causal diamonds

Ref:- M. Saravani, R. Sorkin, Y. Yazdi, Class. Quant. Grav. vol. 31, no. 21, p. 214006, 2014

$$S = b \ln \left[\frac{\ell}{a} \right] + c \qquad b = 0.33277 \text{ and } c = 0.70782$$

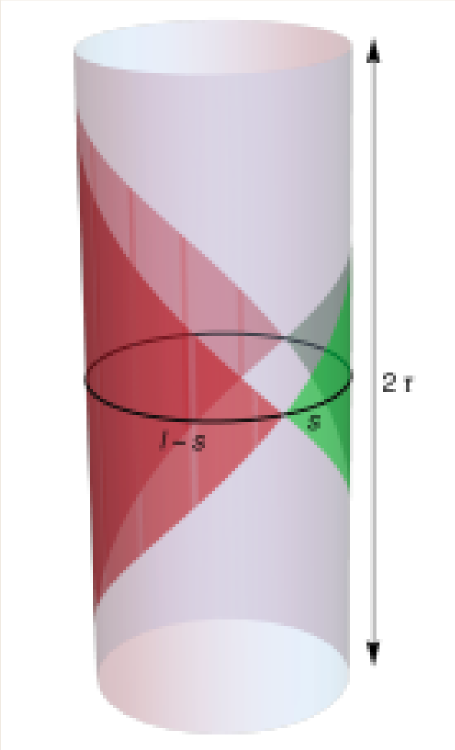


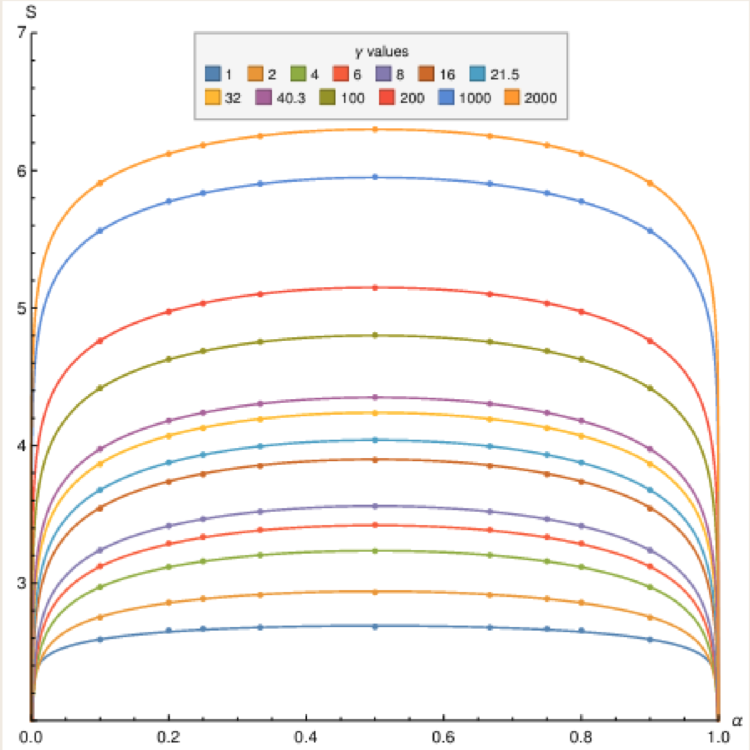
2) Diamond in a cylinder slab

Ref:- A. Mathur, S. Surya, N. X, Phys. Lett. B, vol. 820, p. 136567, 2021

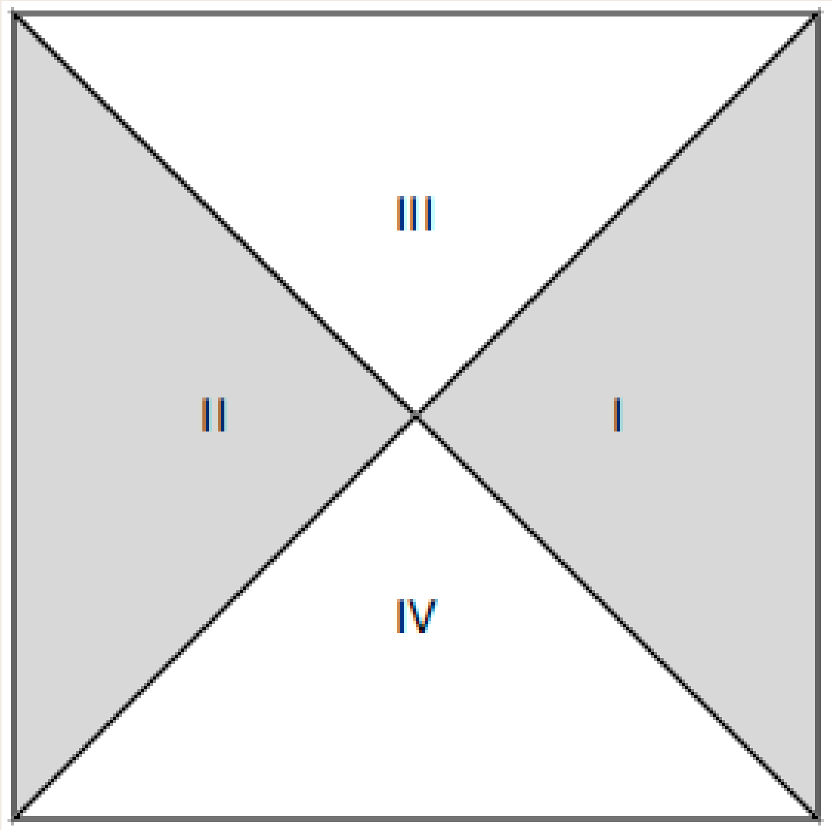
$$\mathcal{S} = \frac{c(\gamma)}{3} \ln\left(\frac{\ell}{\pi \epsilon}\right) + f(\gamma) \ln(\sin(\alpha\pi)) + c_1(\gamma)$$

$$c(\gamma) \sim 1$$
$$f(\gamma) \sim 0.33 + a/\gamma + b/\gamma^2$$
$$c_1(\gamma) \sim a' \log\gamma + b'.$$





Spacetime entanglement entropy in de Sitter



de Sitter spacetime:

$$-X_0^2+\sum_{i=1}^{d}X_i^2=\frac{1}{H^2}$$

Conformal (Poincare) Patch
Region I U III

$$ds^2=\frac{1}{H^2\eta^2}\left(-d\eta^2+dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right)$$

$\eta\in(-\infty,0),\ r\in[0,\infty)$ and $(\theta,\phi)\in\mathbb{S}^2$

Static Patch
Region I

$$ds^2=\frac{1}{H^2}\left(-(1-x^2)dt^2+\frac{dx^2}{1-x^2}+x^2d\Omega_2^2\right)$$

$$x=-\frac{r}{\eta},\quad e^{-t}=\sqrt{\eta^2-r^2}$$

$x\in[0,1),\ t\in\mathbb{R}$

Note: the static patch is non-compact. We can solve for μ as long as the region of interest is static and is spatially compact

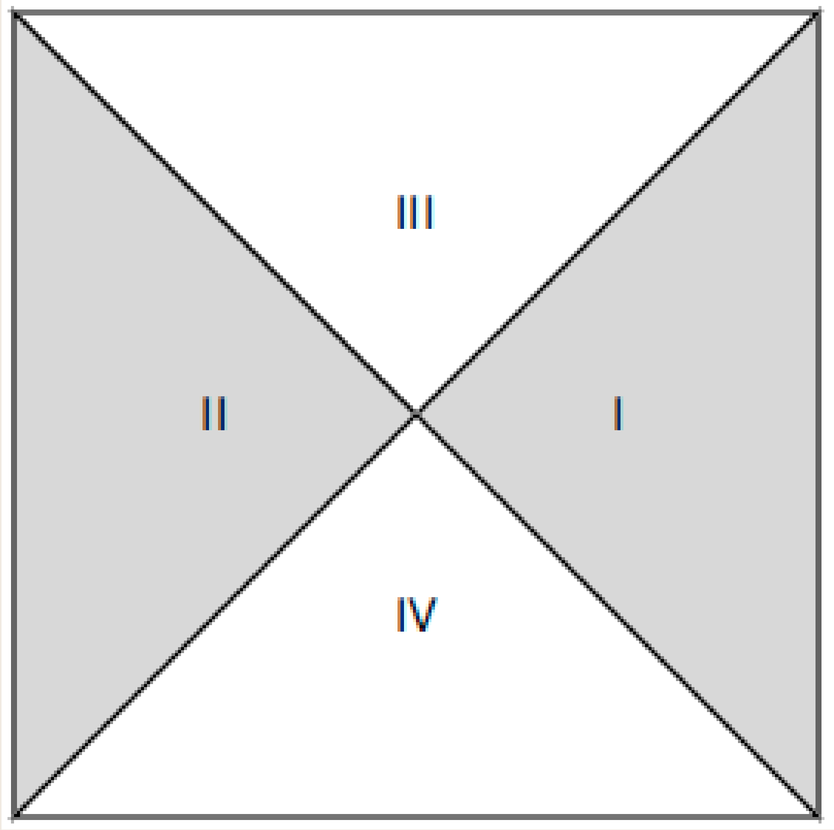
Bunch-Davies modes in I U III

Ref:- T. S. Bunch, P. Davies, Proceedings of the Royal Society of London. A. 360 (1700):117–134, 1978

$$\Phi_{klm} \equiv \varphi_{kl}(\eta, r) Y_{lm}(\theta, \phi)$$

$$\varphi_{kl}(\eta, r) = \frac{H e^{-\frac{i\pi}{2}(l+\frac{1}{2})}}{\sqrt{2k}} (-k\eta)^{\frac{3}{2}} e^{\frac{i\nu\pi}{2}} H_{\nu}^{(1)}(-k\eta) j_l(kr)$$

$$\nu = \sqrt{\frac{9}{4} - \frac{\mathbf{m}^2}{H^2}}$$



static modes in I

Ref:- A. Higuchi, Class. Quant. Grav. vol. 4, p. 721, 1987

$$\Psi_{plm} = \psi_{pl}(t, x) Y_{lm}(\theta, \phi)$$

$$\psi_{pl}(t, x) \equiv \sqrt{2\sinh(\pi p)}\, N_{pl}\, U_{pl}(x) e^{-ipt}, \quad p \in \mathbb{R}^+,$$

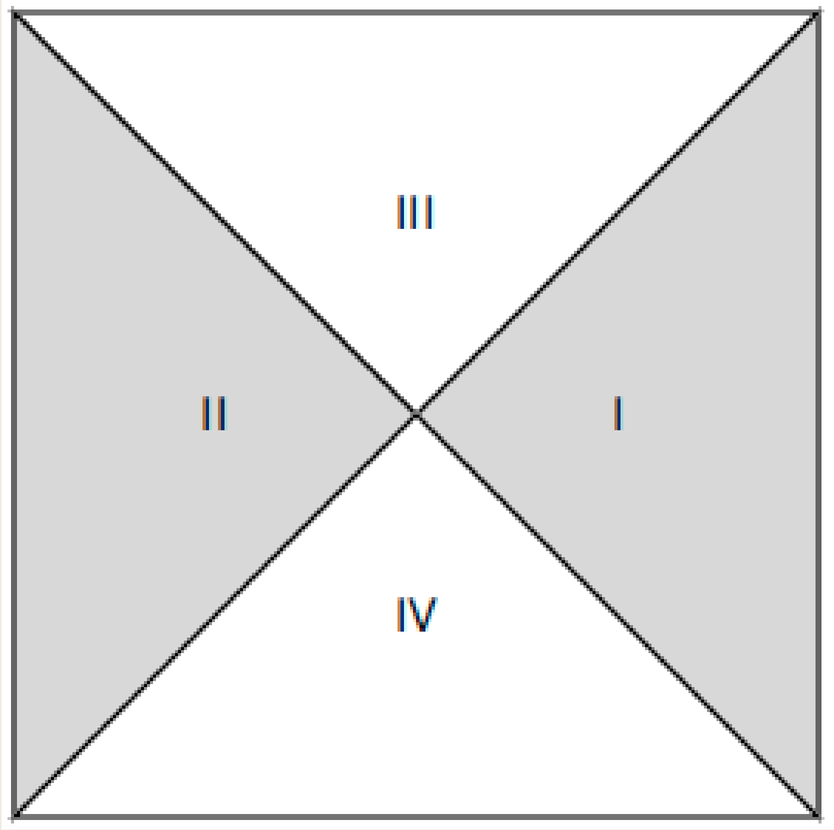
$$N_{pl} = \frac{H}{2\sqrt{2}\pi\Gamma(l+\frac{3}{2})} \Gamma\left(\frac{\frac{3}{2}+l-ip+\nu}{2}\right) \Gamma\left(\frac{\frac{3}{2}+l-ip-\nu}{2}\right),$$

$$U_{pl}(x) = x^l (1-x^2)^{\frac{-ip}{2}} {}_2F_1\left(\frac{\frac{3}{2}+l-ip+\nu}{2}, \frac{\frac{3}{2}+l-ip-\nu}{2}, l+\frac{3}{2}; x^2\right)$$

$\mathcal{L}^2$ orthogonal in region I

$$W(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}(\mathbf{x})\Phi^*_{\mathbf{k}}(\mathbf{x}')$$

$$\Phi_{\mathbf{k}}(\mathbf{x})\Big|_{\mathcal{O}} = \sum_{\mathbf{p}} \left(\alpha_{\mathbf{kp}}\Psi_{\mathbf{p}}(\mathbf{x}) + \beta_{\mathbf{kp}}\Psi^*_{\mathbf{p}}(\mathbf{x})\right)$$ (restriction in region I)



Following the calculations by Higuchi and Yamamoto in Phys. Rev. D 98, 065014 (2018)

$$\alpha_{kp} = \frac{2^{-ip}k^{ip-\frac{1}{2}}}{\sqrt{2\pi(1-e^{-2\pi p})}}, \quad \beta_{kp} = \frac{2^{ip}k^{-ip-\frac{1}{2}}}{\sqrt{2\pi(e^{2\pi p}-1)}}$$

$$W(\mathbf{x},\mathbf{x}')\Big|_{\mathcal{O}} = \sum_{\mathbf{pp}'} \left(A_{\mathbf{pp}'}\Psi_{\mathbf{p}}(\mathbf{x})\Psi^*_{\mathbf{p}'}(\mathbf{x}') + B_{\mathbf{pp}'}\Psi_{\mathbf{p}}(\mathbf{x})\Psi_{\mathbf{p}'}(\mathbf{x}') + C_{\mathbf{pp}'}\Psi^*_{\mathbf{p}}(\mathbf{x})\Psi^*_{\mathbf{p}'}(\mathbf{x}') + D_{\mathbf{pp}'}\Psi^*_{\mathbf{p}}(\mathbf{x})\Psi_{\mathbf{p}'}(\mathbf{x}')\right)$$ --(1)

$$A_{\mathbf{pp}'} \equiv \sum_{\mathbf{k}} \alpha_{\mathbf{kp}}\alpha^*_{\mathbf{kp}'}, \quad B_{\mathbf{pp}'} \equiv \sum_{\mathbf{k}} \alpha_{\mathbf{kp}}\beta^*_{\mathbf{kp}'}, \quad C_{\mathbf{pp}'} \equiv \sum_{\mathbf{k}} \beta_{\mathbf{kp}}\alpha^*_{\mathbf{kp}'}, \quad D_{\mathbf{pp}'} \equiv \sum_{\mathbf{k}} \beta_{\mathbf{kp}}\beta^*_{\mathbf{kp}'}$$

$$A_{pp'} = \frac{\delta(p-p')}{1-e^{-2\pi p}}, \quad D_{pp'} = \frac{\delta(p-p')}{e^{2\pi p}-1} \quad \text{and} \quad B_{pp'} = C_{pp'} = 0$$

$$i\Delta(x;x') = \sum_{\mathbf{p}} \left(\Psi_{\mathbf{p}}(x)\Psi^*_{\mathbf{p}}(x') - \Psi^*_{\mathbf{p}}(x)\Psi_{\mathbf{p}}(x')\right)$$ --(2)

eigenvalues

$$\mu_p^+ = \frac{1}{1 - e^{-2\pi p}} \quad \text{and} \quad \mu_p^- = -\frac{e^{-2\pi p}}{1 - e^{-2\pi p}}$$

mode-wise contribution to the spacetime entanglement entropy

$$\mathcal{S}_p = -\log(1 - e^{-2\pi p}) - \frac{e^{-2\pi p}}{1 - e^{-2\pi p}} \log e^{-2\pi p}$$

(same as the von Neumann entropy obtained by Higuchi and Yamamoto, Phys. Rev. D 98, 065014 (2018))

Can easily be generalised to any static spacetime region with compact spatial hypersurface

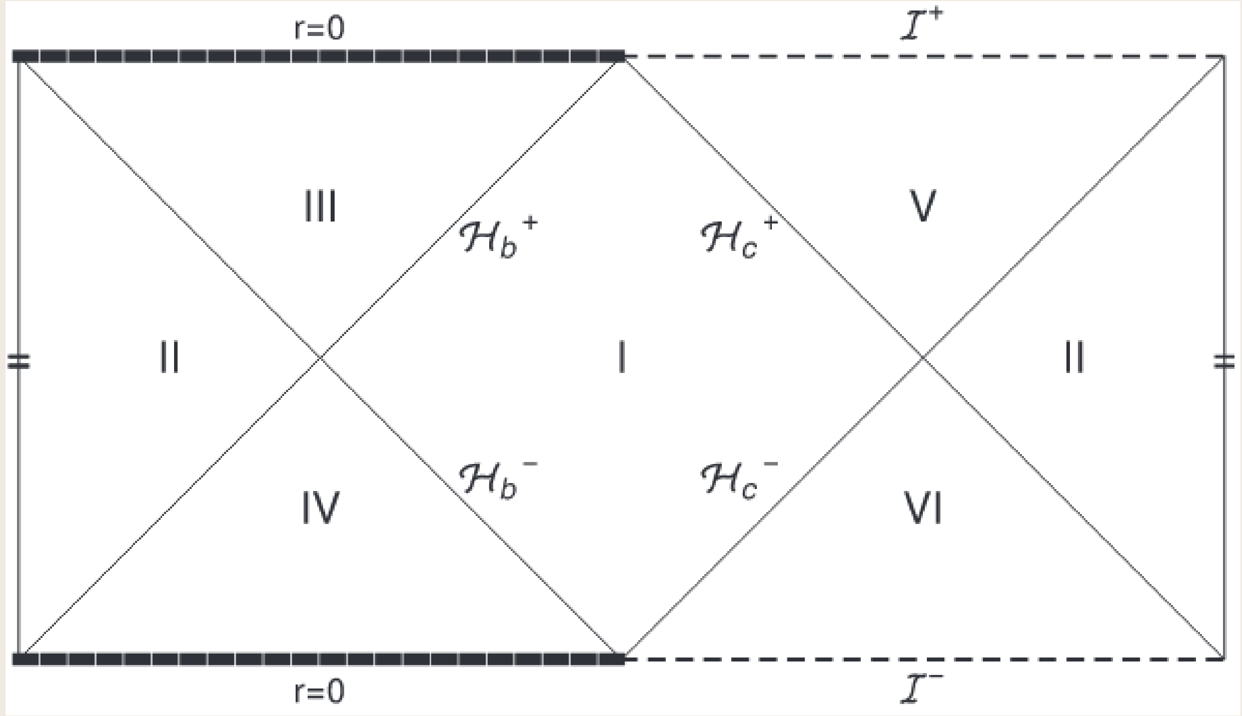
For eg. a static patch of Schwarzschild de Sitter black hole

Spacetime entanglement entropy in Schwarzschild de Sitter black hole

$$\Phi_{klm} = \begin{cases} \frac{1}{\sqrt{4\pi k r_b}} e^{-ikU_b} Y_{lm}(\theta,\phi) & \text{on } \mathcal{H}_b^- \\ 0 & \text{on } \mathcal{H}_c^- \end{cases}, \quad k > 0,$$ Kruskal modes

$$\Psi_{plm} = \begin{cases} \frac{1}{\sqrt{4\pi p r_b}} e^{-ipu} Y_{lm}(\theta,\phi) & \text{on } \mathcal{H}_b^- \\ 0 & \text{on } \mathcal{H}_c^- \end{cases}, \quad p > 0,$$ Static modes

$$U_b = -\kappa_b^{-1} e^{-\kappa_b u} \quad \text{and} \quad V_b = \kappa_b^{-1} e^{\kappa_b v}$$



$$S_{\tilde{p}} = -\log\left(1 - e^{-2\pi\tilde{p}}\right) - \frac{e^{-2\pi\tilde{p}}}{1 - e^{-2\pi\tilde{p}}} \log\left(e^{-2\pi\tilde{p}}\right)$$

$$\tilde{p} \equiv p\kappa^{-1}$$

Discussions

- Sorkin's spacetime entanglement entropy (SSEE) serves as a spacetime definition for the entanglement entropy of a QFT.
- All ingredients involved (W and iΔ) are well defined and have a concrete form for a quantum field system.
- Can be extended to quantum gravity theories like Causal Set Theory.

  R. D. Sorkin, Y. K. Yazdi, Class. Quant. Grav. 35 074004

  S. Surya, Nomaan X, Y. K. Yazdi, Class. Quant. Grav. 38 (2021) 11, 115001
- Agrees with the von Neumann entropy for a system with finite number of dof.

  R. D. Sorkin, J. Phys. Conf. Ser. 484 (2014) 012004

Future Directions

- Although we have shown specific examples of infinite dof, the general construction is still open
- Understanding SSEE in AQFT (eg. Entanglement measures)

  Ref: S. Hollands and K. Sanders