Astrophysically relevant geodesics in Kerr spacetime

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Outline

Contents

Space-time around a rotating black hole

Kerr metric: (*Kerr R. 1963, PRL, 11, 237–238;* Kerr 1963)

$$ds^2 = -(1 - \frac{2M_\bullet r}{\rho^2})dt^2 + \frac{\rho^2}{\Delta}dr^2 + (r^2 + a^2 + \frac{2M_\bullet r a^2 \sin^2\theta}{\rho^2})\sin^2\theta d\phi^2 - \frac{4M_\bullet a r \sin^2\theta}{\rho^2} d\phi\, dt + \rho^2 d\theta^2 \quad (1)$$

$\{t, r, \theta, \phi\}$ - Boyer Lindquist coordinates, $G = c = 1$, geometrical units,

$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta \equiv r^2 - 2M_\bullet r + a^2, \quad a \equiv \mathcal{J}/M_\bullet$ − spin.

- Conserved quantities for timelike geodesics:
  1. Energy per unit rest mass: E, stationary metric.
  2. z-component of the angular momentum: L, axis symmetric metric.
  3. Carter's constant: Q (*Carter 1968, Phys. Rev. D, 174 1559–71; Carter 1968*)
     Separability r and θ components of the Hamilton-Jacobi equation.

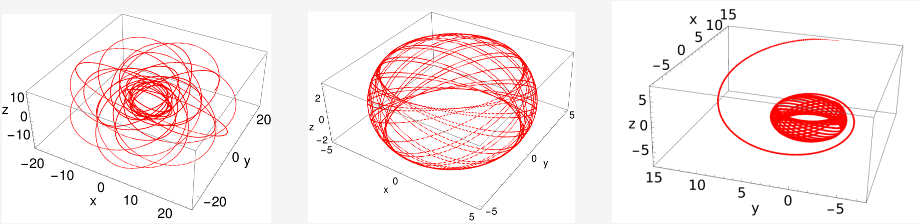
History of timelike Kerr geodesics

- Carter 1968, PRD: Pioneer work on solving Hamilton-Jacobi equation of motion to get quadrature form.
- Wilkins 1972, PRD: Solution for spherical orbits for extreme spin, $a = 1$.
- Bardeen et al 1972, ApJ: Deriving $\{E, L\}$ for circular orbits and ISCO radius.
- Schmidt 2002, CQG: Deriving fundamental orbital frequencies $\{\nu_\phi, \nu_r, \nu_\theta\}$ in quadrature form.
- Fujita & Hikida 2009, CQG: First fully-analytic form of equations of motion.
- **Rana & Mangalam 2019, CQG: Simpler analytic form of equations of motion using eccentricity and inverse-latum rectum of orbit $\{e, \mu\}$, and study of various types of orbits.**
- G. Compère et al 2022, PRD: Detailed study of classification of radial motion.

Contents

3D Kerr trajectories

- Axis symmetry of the spacetime leads to change in plane of the orbit.
- Non-equatorial eccentric bound orbits, spherical orbits, and non-equatorial separatrix orbits. (*Rana P. & A. Mangalam 2019, Class. Quant. Grav. 36, 045009, arXiv:1901.02730, RM19*).

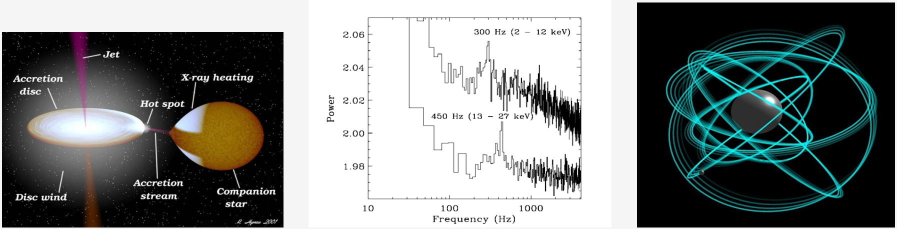


(a) (b) (c)

Figure: 1. (a) A non-equatorial eccentric trajectory, (b) a spherical trajectory, and (c) a separatrix trajectory; RM19.

Motivation to study Kerr orbit dynamics

- QPOs in BHXRB: Study of Quasi-periodic oscillations observed in Fourier power density spectrum of Black hole X-ray binaries, indicating fundamental frequencies of oscillations near a black hole.



(a) (b) (c)

Figure: 2. (a) Black hole X-ray binary, *Courtesy: Web*; (b) Simultaneous QPOs in the PSD of GRO J1655-40 using RXTE data. *Courtesy: T.E.Strohmayer 2001, ApJ, 552, L49.*; (c) A cartoon of an EMRI *Courtesy: Web*.

- GWs from EMRIs: Extreme-mass ratio inspirals follow a series of Kerr orbits before plunging onto the supermassive black hole.

Radial motion

The effective potential for the radial motion

$$\underbrace{\boxed{\frac{(E^2-1)}{2}}}_{\text{Total energy}} = \underbrace{\boxed{\frac{1\rho^4}{2r^4}\left(\frac{dr}{d\tau}\right)^2}}_{\text{Kinetic energy}} - \underbrace{\boxed{\frac{1}{r} + \frac{L^2 - a^2\left(E^2-1\right) + Q}{2r^2} - \frac{(L-aE)^2 + Q}{r^3} + \frac{a^2 Q}{2r^4}}}_{\text{Effective potential}} \quad (2)$$

Roots of the radial equation using Ferrari's method (*G. Cardano, The rules of algebra. Translated by T. R. Witmer 1968; Cardano 1968*)

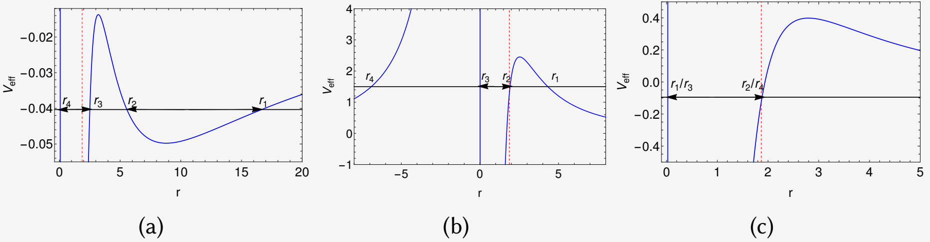


Figure: 3. Radial effective potentials for various radial motions.

Trajectory solutions (RM19)

- Equations for time-like geodesics (*Carter 1968*, *Schmidt 2002*)

$$\phi - \phi_0 = -\frac{1}{2}\int_{r_0}^{r} \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial L} dr' - \frac{1}{2}\int_{\theta_0}^{\theta} \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial L} d\theta' = -\frac{1}{2}I_1 - \frac{1}{2}H_1, \tag{3a}$$

$$t - t_0 = \frac{1}{2}\int_{r_0}^{r} \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial E} dr' + \frac{1}{2}\int_{\theta_0}^{\theta} \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial E} d\theta' = \frac{1}{2}I_2 + \frac{1}{2}H_2, \tag{3b}$$

$$\int_{r_0}^{r} \frac{dr'}{\sqrt{R}} = \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\Theta}} \Rightarrow I_8 = H_3, \tag{3c}$$

where $R(r) \geq 0$ and $\Theta(\theta) \geq 0$.

- Substituting $r_1 = 1/\left[\mu\left(1-e\right)\right]$ and $r_2 = 1/\left[\mu\left(1+e\right)\right]$ removes the singularity of integrands at turning points:

$$R = \left(E^2 - 1\right)\left(r - r_1\right)\left(r - r_2\right)\left(r - r_3\right)\left(r - r_4\right). \tag{4}$$

- (e, μ, a, Q) parameter space gives simpler analytic compact forms of the integrals I_1, I_2, I_8, H_1, H_2, and H_3.

Contents

Analytic forms for trajectory solutions (RM19)

$$\phi - \phi_0 = -\frac{1}{2}I_1 - \frac{1}{2}H_1 = \frac{1}{2}\left[C_3 I_3\left(\alpha, e, \mu, a, Q\right) + C_4 I_4\left(\alpha, e, \mu, a, Q\right) - H_1\left(\theta, \theta_0, e, \mu, a, Q\right)\right], \tag{5a}$$

$$\begin{aligned} t - t_0 = {} & \frac{1}{2}I_2 + \frac{1}{2}H_2 = \frac{1}{2}\left[C_5 I_5\left(\alpha, e, \mu, a, Q\right) + C_6 I_6\left(\alpha, e, \mu, a, Q\right) + C_7 I_3\left(\alpha, e, \mu, a, Q\right)\right. \\ & \left. + C_8 I_4\left(\alpha, e, \mu, a, Q\right) + H_2\left(\theta, \theta_0, e, \mu, a, Q\right)\right], \end{aligned} \tag{5b}$$

$$I_8\left(\alpha, e, \mu, a, Q\right) = H_3\left(\theta, \theta_0, e, \mu, a, Q\right), \tag{5c}$$

Integrals	Analytic forms of the radial integrals
I_3	$\frac{1}{\sqrt{1-m^2}\left(m^2+p_2^2\right)}\left[m^2 F\left(\alpha, k^2\right) + p_2^2 \Pi\left(\frac{-p_2^2 - m^2}{1-m^2}, \alpha, k^2\right)\right]$
I_4	$\frac{1}{\sqrt{1-m^2}\left(m^2+p_3^2\right)}\left[m^2 F\left(\alpha, k^2\right) + p_3^2 \Pi\left(\frac{-p_3^2 - m^2}{1-m^2}, \alpha, k^2\right)\right]$
I_5	$\frac{1}{\sqrt{1-m^2}\left(m^2+p_1^2\right)^2}\left[m^4 F\left(\alpha, k^2\right) + 2p_1^2 m^2 \Pi\left(s^2, \alpha, k^2\right) + p_1^4 I_7\left(\alpha, e, \mu, a, Q\right)\right]$
I_6	$\frac{1}{\sqrt{1-m^2}\left(m^2+p_1^2\right)}\left[m^2 F\left(\alpha, k^2\right) + p_1^2 \Pi\left(s^2, \alpha, k^2\right)\right]$
I_7	$\frac{s^4 \sin\alpha\cos\alpha\sqrt{1-k^2\sin^2\alpha}}{2\left(1-s^2\right)\left(k^2-s^2\right)\left(1-s^2\sin^2\alpha\right)} + \frac{\left[s^4 - 2s^2\left(1+k^2\right)+3k^2\right]}{2\left(1-s^2\right)\left(k^2-s^2\right)}\Pi\left(s^2, \alpha, k^2\right) - \frac{1}{2\left(1-s^2\right)}F\left(\alpha, k^2\right) - \frac{s^2}{2\left(1-s^2\right)\left(k^2-s^2\right)}K\left(\alpha, k^2\right)$
I_8	$\frac{2\mu\left(1-e^2\right)}{\sqrt{C - A + \sqrt{B^2 - 4AC}}}F\left(\alpha, k^2\right)$

Analytic forms for trajectory solutions (RM19)

Integrals	Analytic forms of the θ integrals
H_1	$\frac{2L}{z_+ a\sqrt{1-E^2}}\left\{\Pi\left(z_-^2, \arcsin\left(\frac{\cos\theta}{z_-}\right), \frac{z_-^2}{z_+^2}\right) - \Pi\left(z_-^2, \arcsin\left(\frac{\cos\theta_0}{z_-}\right), \frac{z_-^2}{z_+^2}\right) - F\left(\arcsin\left(\frac{\cos\theta}{z_-}\right), \frac{z_-^2}{z_+^2}\right) + F\left(\arcsin\left(\frac{\cos\theta_0}{z_-}\right), \frac{z_-^2}{z_+^2}\right)\right\}$
H_2	$\frac{2Eaz_+}{\sqrt{1-E^2}}\left\{K\left(\arcsin\left(\frac{\cos\theta}{z_-}\right), \frac{z_-^2}{z_+^2}\right) - K\left(\arcsin\left(\frac{\cos\theta_0}{z_-}\right), \frac{z_-^2}{z_+^2}\right) - F\left(\arcsin\left(\frac{\cos\theta}{z_-}\right), \frac{z_-^2}{z_+^2}\right) + F\left(\arcsin\left(\frac{\cos\theta_0}{z_-}\right), \frac{z_-^2}{z_+^2}\right)\right\}$
H_3	$\frac{1}{a\sqrt{1-E^2}z_+}\left\{F\left(\arcsin\left(\frac{\cos\theta_0}{z_-}\right), \frac{z_-^2}{z_+^2}\right) - F\left(\arcsin\left(\frac{\cos\theta}{z_-}\right), \frac{z_-^2}{z_+^2}\right)\right\}$

Table: 1. Analytic expressions for integrals involved in the equations of motion.

where m^2, p_1^2, p_2^2, p_3^2, k^2, s^2, A, B, C, z_-, z_+, and C_3-C_8 are functions of parameters (e, μ, a, Q).

- Advantage of novel trajectory solution: Removes the singularity at the turning points of orbits and provides faster computation.
- Standard Elliptic integrals: $F\left(\alpha, k^2\right)$, $\mathrm{K}\left(\alpha, k^2\right)$, $\Pi\left(s^2, \alpha, k^2\right)$

Bound orbit conditions in (e, μ, a, Q) space (RM19)

- Condition for the real integrals:

$$\left[\mu^3 a^2 Q (1+e)^2 + \mu^2 \left(\mu a^2 Q - x^2 - Q\right) \times (3-e)(1+e) + 1\right] \geq 0. \quad \text{(6a)}$$

- Periastron point is outside horizon:

$$\left[\mu (1+e) \left(1 + \sqrt{1-a^2}\right)\right] < 1. \quad \text{(6b)}$$

- Bound orbit between outermost roots of the effective potential:

$$E(e, \mu, a, Q) < 1. \quad \text{(6c)}$$

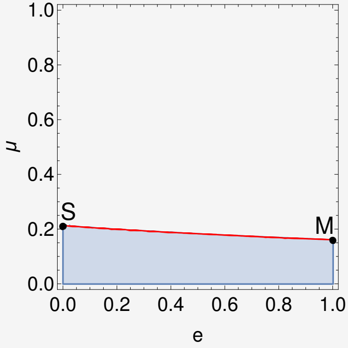


Figure 4. The bound orbit region in the (e, μ) plane for $a = 0.5$ and $Q = 5$. The red curve represents separatrix orbits. S represents ISSO, and M represents MBSO.

ISSO and MBSO

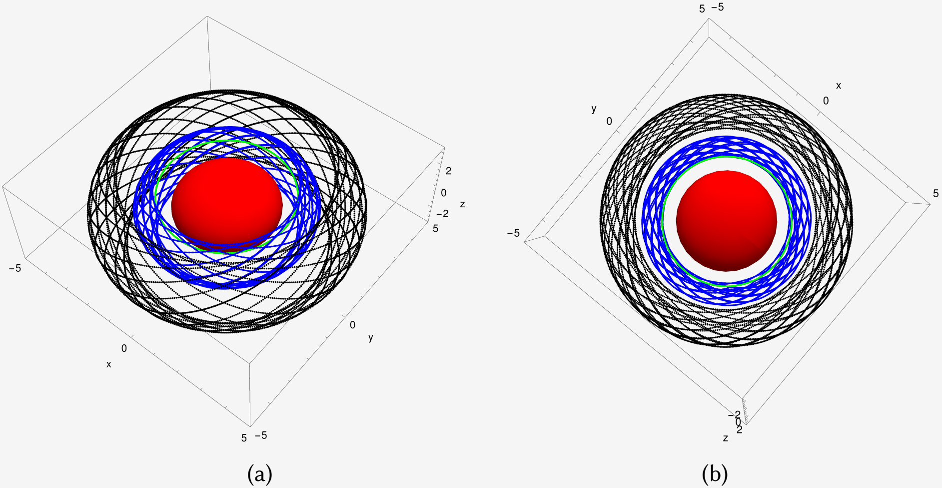
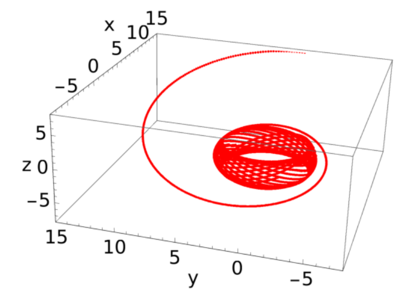
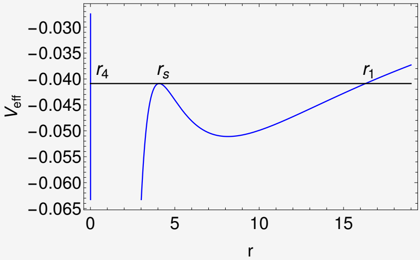


Figure: 5. The horizon radius (r_+, red), light radius [green ring], MBSO (blue), and ISSO (black) (embedded in flat space), (a) side view, (b) top view, are shown, where $\{a = 0.5, Q = 4\}$.

Separatrix trajectories (RM19)



(a)



(b)

Figure: 7. A Separatrix trajectory with $\{e = 0.6,\ r_p = 4.075 R_g,\ a = 0.2,\ Q = 3\}$ satisfying Eq. (8).

- Every inspiraling orbit passes through a separatrix orbit on the transition from bound to plunge. Hence, they are important in the study of inspiralling bodies, for example, study of GW emission from extreme-mass ratio inspirals (EMRIs).

Separatrix trajectories (RM19)

- Position of separatrix in the parameter space:

$$e_s(r_s, a, Q) = \frac{4 + a' r_s + \sqrt{\left(r_s a' + 2\right)^2 - 4d' r_s^4}}{-a' r_s - \sqrt{\left(r_s a' + 2\right)^2 - 4d' r_s^4}}, \tag{7a}$$

$$\mu_s(r_s, a, Q) = \frac{1}{4r_s}\left[-a' r_s - \sqrt{\left(r_s a' + 2\right)^2 - 4d' r_s^4}\right]; \tag{7b}$$

where a' and d' are functions of (r_s, a, Q). These formulae reduce to the equatorial case, $Q = 0$, previously derived by *Levin & Perez-Giz, 2009, PRD, 79(12), 124013*.

- Trajectory solutions: Using equation

$$\left[\mu^3 a^2 Q (1+e)^2 + \mu^2 \left(\mu a^2 Q - x^2 - Q\right)(3-e)(1+e) + 1\right] = 0, \tag{8}$$

we obtained trajectory solutions in terms of the logarithmic and trigonometric functions.

Equatorial eccentric orbits ($e \neq 0$, $Q = 0$) and spherical orbits ($e = 0$, $Q \neq 0$) (RM19)

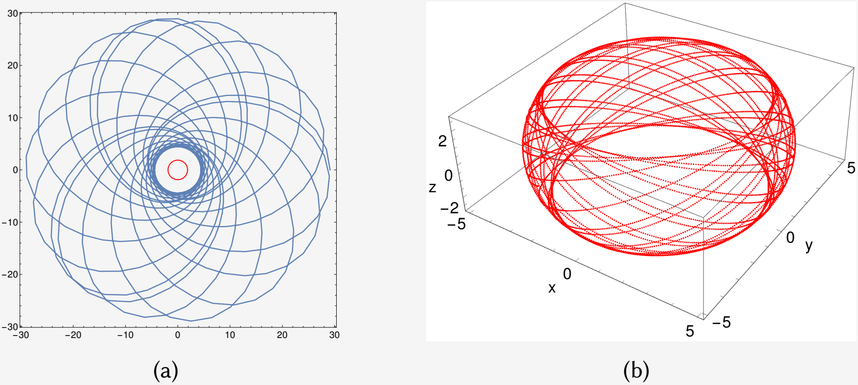


Figure: 6. (a) An equatorial eccentric trajectory, and (b) a spherical orbit near a rotating black hole.

Energy and Angular momentum of spherical orbits (RM19)

- General expression for E and L for spherical orbits as functions of (r_s, a, Q):

$$E(r_s, a, Q) = \frac{\left\{ \begin{array}{r} 2a^4Q + (r_s-3)(r_s-2)^2 r_s^4 - a^2 r_s \left[r_s^2 (3r_s - 5) + Q(r_s(r_s-4)+5) \right] \\ -2a\left[r_s(r_s-2) + a^2 \right] \sqrt{a^2Q^2 - r_s^3 Q(r_s-3) + r_s^5} \end{array} \right\}^{1/2}}{r_s^2 \left[r_s (r_s-3)^2 - 4a^2 \right]^{1/2}}, \tag{9a}$$

$$x(r_s, a, Q) = \frac{\left\{ \begin{array}{r} -2a^4Q + r_s^2 (r_s-3)\left[r_s^2 - (r_s-3)Q \right] + a^2 r_s \left(r_s^3 + r_s^2 - 2Qr_s + 8Q \right) \\ -2a\left[r_s(r_s-2) + a^2 \right] \sqrt{a^2Q^2 - r_s^3 Q(r_s-3) + r_s^5} \end{array} \right\}^{1/2}}{r_s^{1/2} \left[r_s (r_s-3)^2 - 4a^2 \right]^{1/2}}, \tag{9b}$$

and

$$L(r_s, a, Q) = x(r_s, a, Q) + aE(r_s, a, Q). \tag{9c}$$

- These expressions reduce to the standard formulae of (E, L) for equatorial circular orbits, $Q = 0$ (*Bardeen et. al. 1972, ApJ., 178, 347-370.*).

Fundamental frequencies (RM19)

- We also derive the fundamental frequencies (ν_ϕ, ν_r, ν_θ) using a long time average method.
- Exact expressions of ν_ϕ, ν_r, and ν_θ for non-equatorial eccentric trajectories:

$$\nu_\phi = \frac{c^3 \left\{ \left[-I_1\left(\frac{\pi}{2}, e, \mu, a, Q\right) - 2LI_8\left(\frac{\pi}{2}, e, \mu, a, Q\right) \right] F\left(\frac{\pi}{2}, \frac{z_-^2}{z_+^2}\right) + 2LI_8\left(\frac{\pi}{2}, e, \mu, a, Q\right) \Pi\left(z_-^2, \frac{\pi}{2}, \frac{z_-^2}{z_+^2}\right) \right\}}{2\pi G M_\bullet f(e, \mu, a, Q)}, \tag{10a}$$

$$\nu_r = \frac{c^3 F\left(\frac{\pi}{2}, \frac{z_-^2}{z_+^2}\right)}{G M_\bullet f(e, \mu, a, Q)}, \qquad \nu_\theta = \frac{c^3 a\sqrt{1-E^2}\, z_+ I_8\left(\frac{\pi}{2}, e, \mu, a, Q\right)}{2 G M_\bullet f(e, \mu, a, Q)}, \tag{10b}$$

- To summarize:

  Our analytic results have made advancements to various orbital dynamics study done by Bardeen et. al. 1972, Schmidt 2002, Levin & Perez-Giz 2009, and Fujita & Hikida 2009 in the Kerr space-time.

Contents

Summary

	Results presented in the previous studies	New results
1.	*Bardeen1972*: Analytic expressions of E and L for equatorial circular orbits.	**Spherical orbits**: Analytic expressions for E and L and important radii - ISSO, MBSO and general spherical orbits with $Q \neq 0$.
2.	*Schmidt2002*: Expressions for fundamental frequencies, $(\nu_\phi, \nu_r, \nu_\theta)$, of general trajectories given in terms of quadratures using action-angle variables.	**Frequencies**: t, ϕ, $r-\theta$ integrals are solved analytically and explicit expressions for $(\nu_\phi, \nu_r, \nu_\theta)$ are presented for $Q \neq 0$ using long-time average method.
3.	*Fujita2009*: Analytic trajectory solutions are presented as a function of the variable Mino time, λ, for $Q \neq 0$, where input variables are E, L_z, Q and a.	**Trajectory**: Analytic solutions are presented which provide faster computation using (e, μ, a, Q) space. - Bound orbit conditions in $\{e, \mu, a, Q\}$ and $\{E, L, a, Q\}$ spaces are derived. - Simpler trajectory solutions are obtained for equatorial eccentric and spherical orbits. - Translation relations are obtained between $\{E, L, a, Q\}$ and $\{e, \mu, a, Q\}$ parameter spaces.
4.	*Levin2009*: Expressions for the eccentricity, e_s, and inverse latus-rectum, μ_s, of equatorial separatrix orbits and the trajectory solutions $(Q = 0)$ are presented.	**Separatrix**: Expressions for e_s and μ_s and trajectory solutions are presented for general separatrix orbits with $Q \neq 0$.

Table: 2. A list of novel formulae and advancement in the theoretical results.

Applications

- *Rana, P. & Mangalam, A., 2020a, "A Geometric Origin for Quasi-periodic Oscillations in Black Hole X-Ray Binaries", ApJ., 903(2), 121, arXiv:2009.01832; RM20a.*

- *Rana, P. & Mangalam, A., 2020b, "A Relativistic Orbit Model for Temporal Properties of AGN", Galaxies, 8(3), 67, arXiv:2009.03061; RM20b.*