Covariant formulation of Generalised Uncertainty Principle (GUP)

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Outline

- Introduction to GUP
- Well-Known GUP models
- A covariant approach based on Normal coordinates
- Geometry of momentum space
- Our GUP formalism
- Conclusion

Introduction to GUP

- Heisenberg uncertainty principle tells us, there should be a fundamental limit for the measurement accuracy, with which certain pairs of physical observables, such as the position and momentum and energy and time, can not be measured, simultaneously at arbitrary accuracy.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

- What if someone probes a high energy limit (high momentum)? Do we not need to consider gravitational effects since such high energies affect the background space-time?

Of course, yes

Well-known GUP theories

- Heisenberg algebra gets modified at large momentum, with the Planck scale as the characteristic scale at which the modification occurs.

$$[x^a, p_b] = i\hbar\Theta^a_b(x, p)$$

- Different approaches assume different deformation of algebra, some of the famous approaches are given below.

$$\frac{-i}{\hbar}[X, P] = 1 + \beta p^2 \qquad \text{(Kempf, Mangano, Mann)}$$

$$= 1 - \alpha p + \alpha^2 p^2 \qquad \text{(A. F. Ali, S. Das, E. C. Vagenas)}$$

$$= \frac{1}{1 - \beta p^2} \qquad \text{(Pedram)}$$

- In general, the modification of Heisenberg algebra leads to non-commutativity in position or momentum or both.

If

$$\Theta_{ab} \to \Theta_{ab}(p) \longrightarrow [p_a, p_b] = 0$$

$$[x^a, x^b] \neq 0$$

If

$$\Theta_{ab} \to \Theta_{ab}(x) \longrightarrow [x^a, x^b] = 0$$

$$[p_a, p_b] \neq 0$$

And

$$\Theta_{ab} \to \Theta_{ab}(x, p) \longrightarrow [x^a, x^b] \neq 0$$

$$[p_a, p_b] \neq 0$$

A covariant approach to GUP

- The generalisation and interpretation of the Heisenberg algebra in a fully relativistic theory, which is also invariant under general coordinate transformations is not straightforward.

- The key issue is trivially evident from the explicit appearance of $\hat{x}^i$ in the commutator relationships, which spoils any hope of general covariance.

- This is a big difficulty since general covariance is also the first step towards generalization a theory to a curved background space, or spacetime.

- Therefore, a curved space(-time) generalisation of the Heisenberg algebra requires a covariant definition.

$\longrightarrow$ ***Normal coordinates***

Normal Coordinates

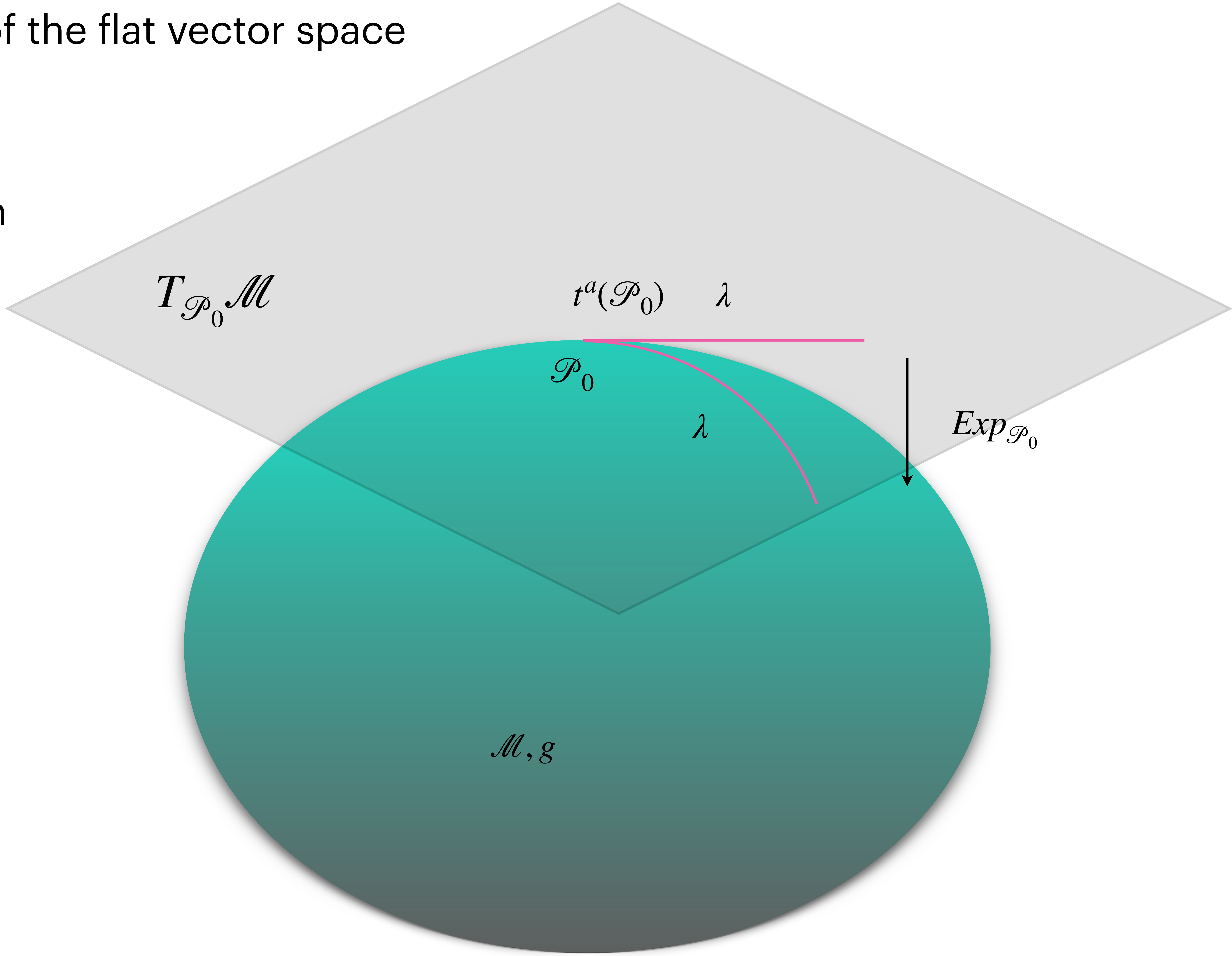
- The Riemannian exponential map establishes a local parametrization of a small region around a location $\mathscr{P}_0 \in \mathscr{M}$ in terms of coordinates of the flat vector space $T_{\mathscr{P}_0}\mathscr{M}$.

- And this is referred to as representing the manifold in normal coordinates Φ^a.

$$\Phi^a = -\eta^{ab} e_b^{i'} \nabla_{i'} \sigma$$

$$\Phi^a = \lambda t^a(\mathscr{P}_0)$$

$$\eta_{ab} \Phi^a \Phi^b = 2\sigma(x', x)$$



Exponential map

Normal Coordinates

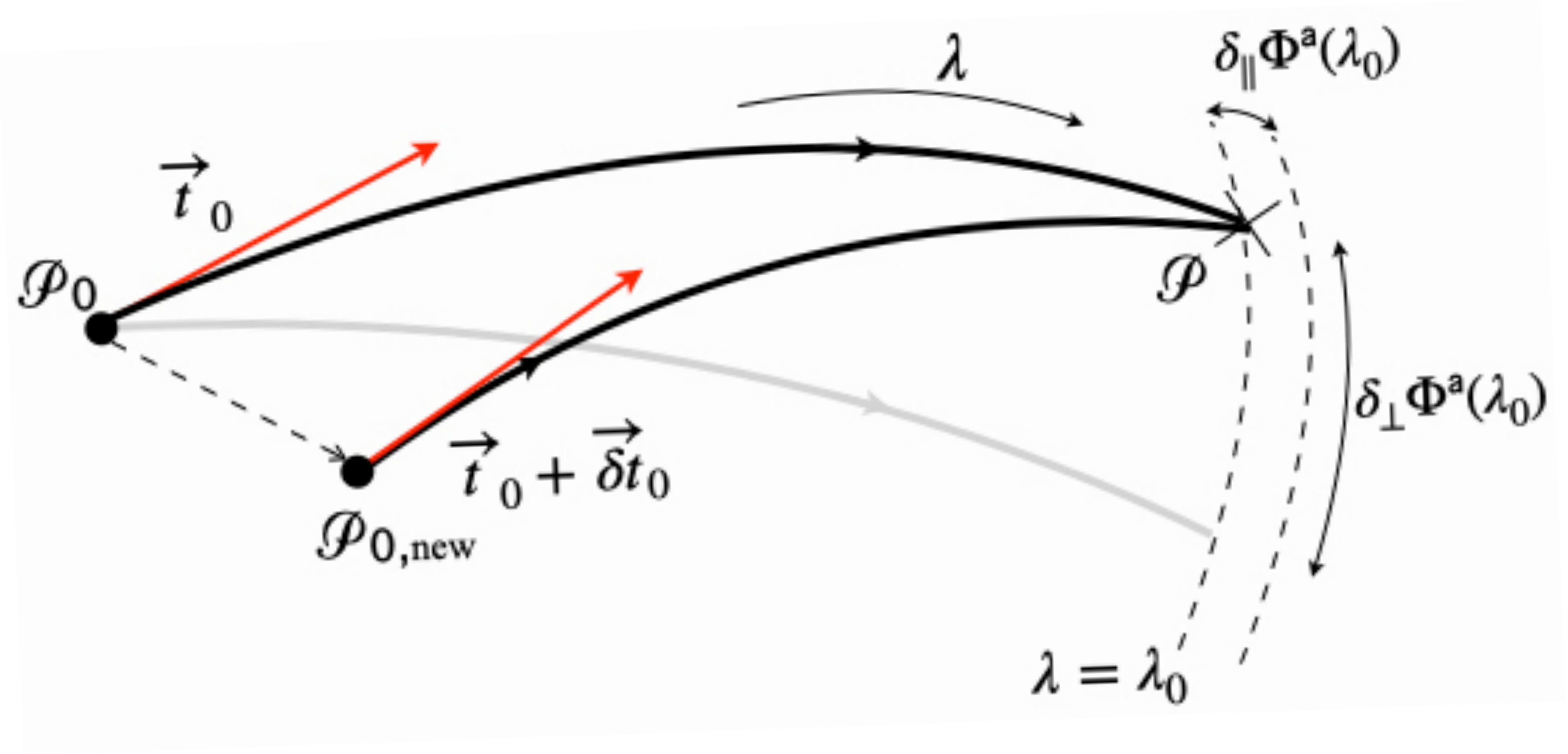
$$\Phi^a(\mathscr{P}) = \lambda t^a\left(\mathscr{P}_0\right)$$

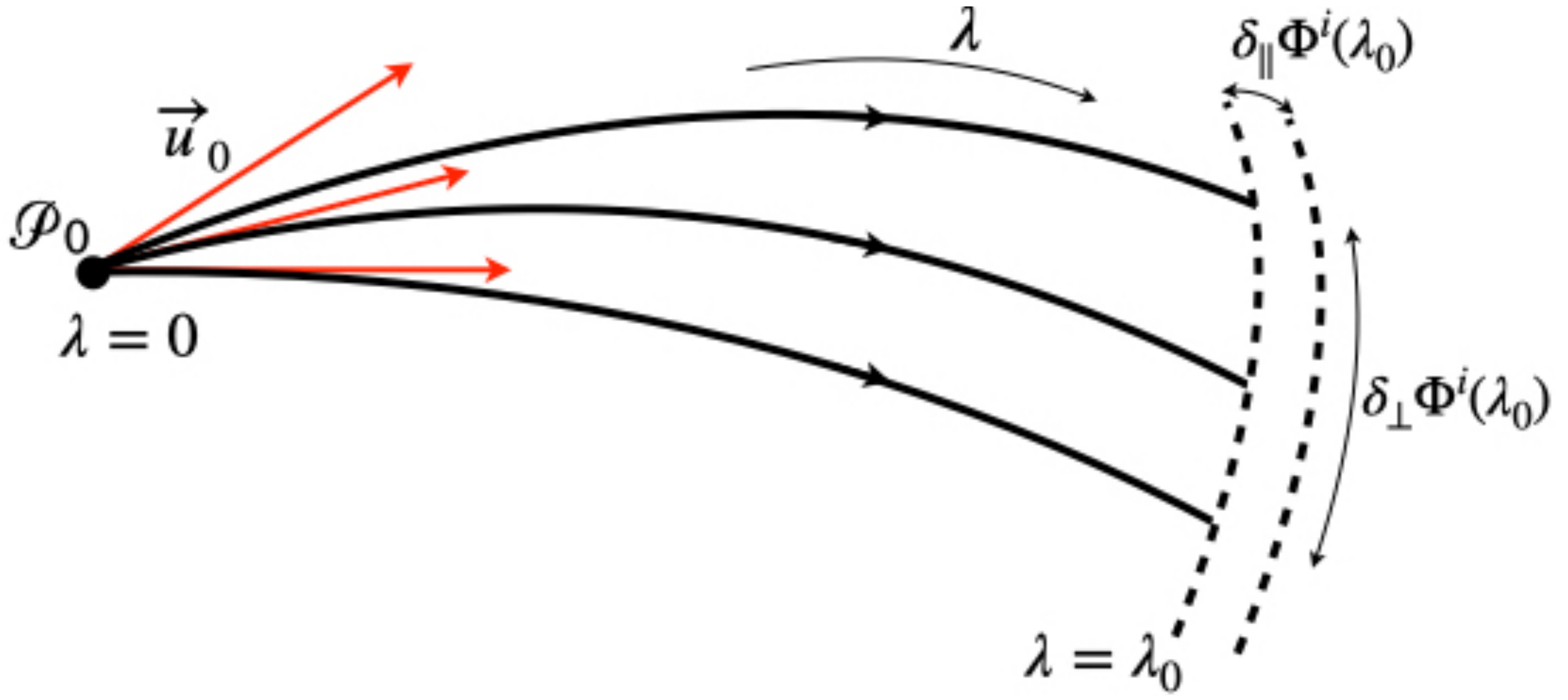
- The variation of the Normal coordinates gives us,

$$\delta\Phi^a(\mathscr{P}) = \lambda[\delta t^a\left(\mathscr{P}_0\right)]_{\lambda\text{ fixed}} + t^a\left(\mathscr{P}_0\right)[\delta\lambda]_{t^a\text{ fixed}}$$

$$[\delta t^a\left(\mathscr{P}_0\right)]_{\lambda\text{ fixed}} = K^a{}_b\varepsilon^b$$

$$\delta\Phi^a(\mathscr{P}) = \left(\lambda K^a{}_b - t^a t_b\right)\varepsilon^b\,.$$





Normal Coordinates

- When $\mathscr{M}$ represents the spacetime manifold, the normal coordinates $\Phi^a(\mathscr{P}) \equiv x^a$, and the variation of normal coordinates suggest the commutator between x^a and the operator p_b generating the shift of origin by ε^b:

$$[x^a, p_b] \stackrel{\text{def}}{:=} i\hbar \left(\lambda K^a{}_b - t^a t_b\right)$$

- In flat spacetime, it follows from the geometry of the equi-geodesic surfaces,

$$\lambda K^a{}_b - t^a t_b = \delta^a_b \,.$$

$$[x^a, p_b] = i\hbar\delta^a_b$$

- We will now turn to the case when $\mathscr{M}$ represents the momentum space, and look for the kind of deformations that the geometry of the momentum space can produce in the commutators.

- We need information about the equi-geodesic surface and its extrinsic curvature in the momentum space, and for this, we need to first characterise the geometry of the momentum space.

- The motivation to construct the curved momentum space can be taken from relative velocities of two points with velocities $\overrightarrow{v}$ and $\overrightarrow{v} + \overrightarrow{dv}$

$$dl_v^2 = \frac{dv^2}{1-v^2} + \frac{v^2}{1-v^2}(d\theta^2 + sin^2\theta \, . \, d\phi^2)$$

$$= d\chi^2 + sinh^2\chi d\Omega^2$$

L. D. Landau, E. M. Lifshitz, Classical theory of fields

Geometry of momentum space

- We write the four dimensional line element by demanding the following conditions,

1. The four momentum geometry to be Lorentzian.
2. For points in momentum space that have zero relative velocity i.e. $dl_{\rm rel} = 0$, the metric gives the difference in rest masses (or rest energies) associated with the corresponding momenta.

$$dl^2_{\rm rel} = 0 \longrightarrow dl^2 = -\,dm^2$$

- Considering all the points, we get the momentum space metric as,

$$dl^2 = -\,dm^2 - p^2 dl^2_{\rm rel}$$

$$dl^2 = -\frac{p^2 F'^2}{F} dm^2 + \mu^2 dl^2_{\rm rel}$$

Geometry of momentum space

$$dl^2 = -\frac{p^2F'^2}{F}dm^2 + \mu^2 dl^2_{\text{rel}}$$

- Where $F(p^2) = -m^2$ denotes different dispersion relations

$$p^2 = -m^2 \longrightarrow$$ Flat Minkowski metric in hyperbolic coordinate

- Modified dispersion relations $F(p^2) = -m^2$ correspond to curved momentum space, with the following curvature,

$$R = \frac{6F}{\mu^2}\left\{-\frac{2F''}{F'^3} + \left(\frac{1}{F} + \frac{1}{\mu^2 F'}\right)\left(1 - \frac{1}{F'}\right)\right\}$$

Our GUP Formalism

- In terms of the dispersion relation $F(p^2) = -m^2$, the commutators are given by

$$[\mathrm{x}^a, \mathrm{p}_b] := i\hbar \left\{ \delta^a_b + \left(\frac{F(p^2)}{p^2 F'(p^2)} - 1 \right) h^a{}_b \right\}$$

- By the use of the normal coordinates, and recognise that 'm' represents the geodesic length (from the origin) in the momentum space metric. The above expression then becomes:

$$[\mathrm{x}^a, \mathrm{p}_b] = i\hbar \left\{ \overbrace{\frac{F}{p^2 F'}}^{G_1} \delta^a_b - \frac{1}{F} \underbrace{\left(\frac{F}{p^2 F'} - 1 \right)}_{G_2} p^a p_\mathrm{b} \right\}$$

Our GUP Formalism

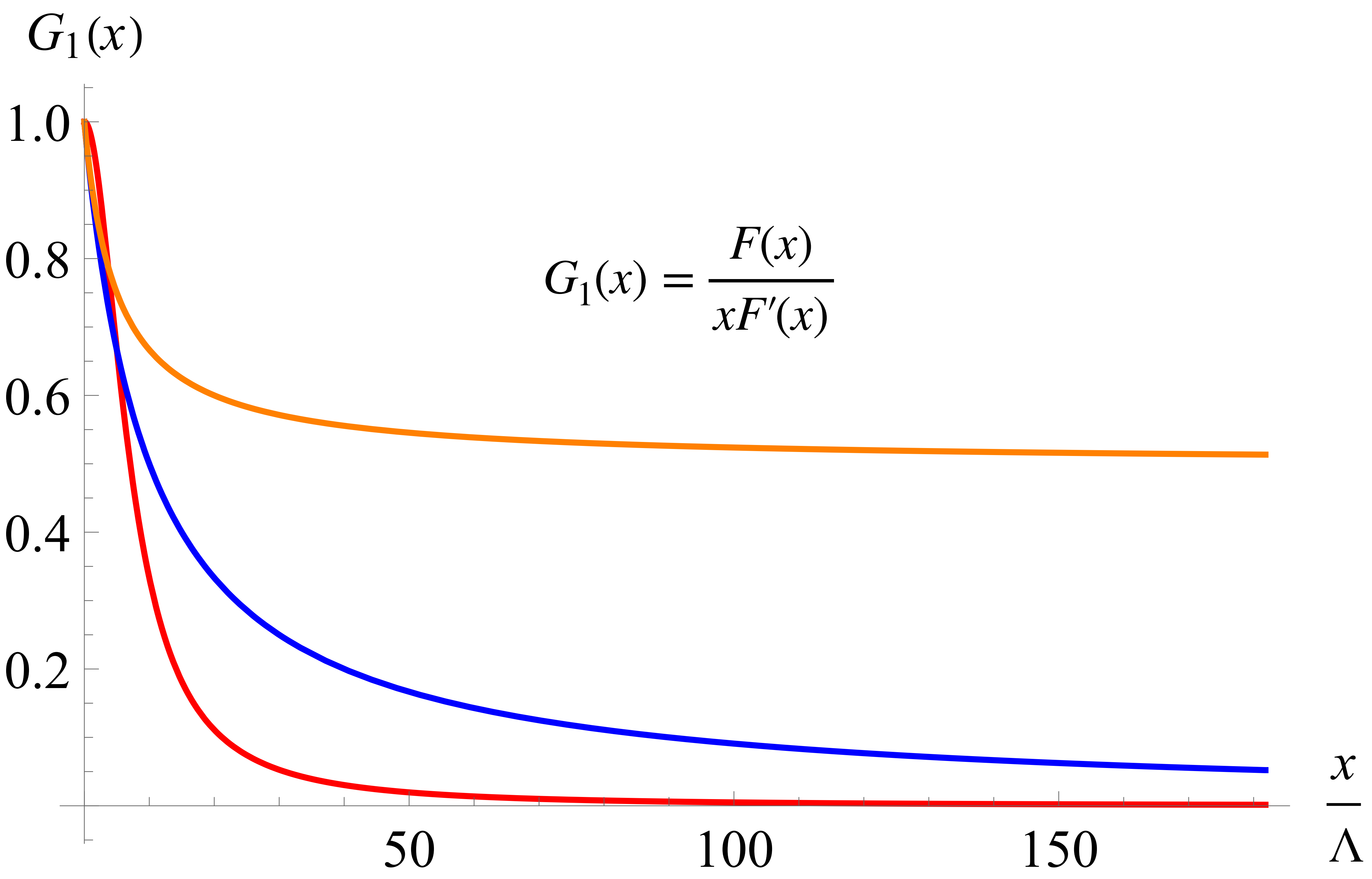
- The remaining commutators then follow from a straightforward application of the Jacobi identity

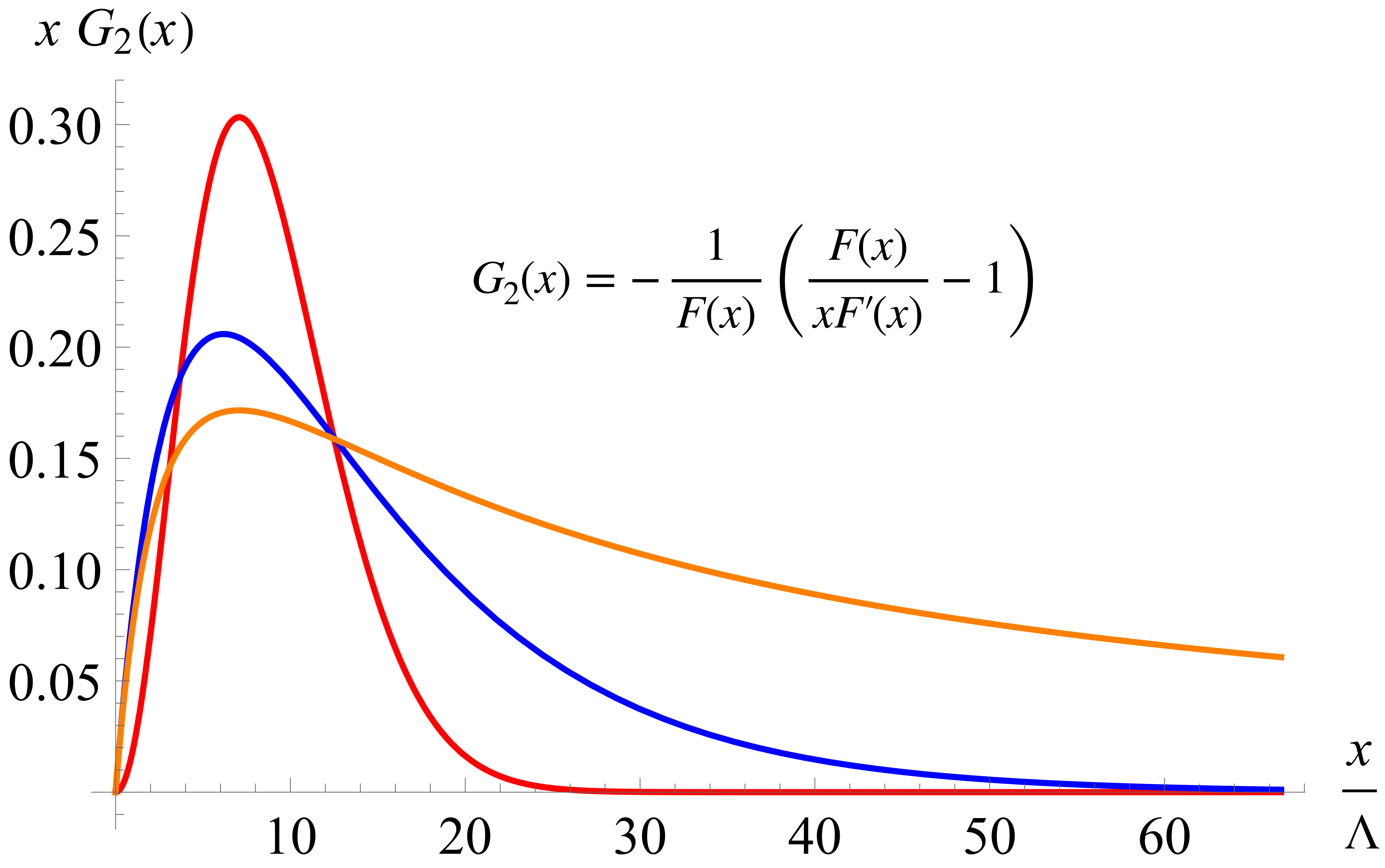
$$[p_a, p_b] = 0$$

$$[x^a, x^b] = 2i\hbar\left(G_2 - 2G_1' - \frac{2G_1'G_2p^2}{G_1}\right)x^{[a}p^{b]}$$

- Evidently, the modified dispersion relation introduces a non-commutativity in normal coordinates.

Our GUP Formalism





$$F(x) = x + x\left(\exp(x/\Lambda)^2 - 1\right)$$

$$F(x) = x\left(1 + (x/\Lambda)^2\right)$$

$$F(x) = x\exp(x/\Lambda)$$

Conclusion

- We have presented a geometric formalism for the generalised uncertainty principle which is covariant and connects features of the underlying geometry with the deformation of canonical commutation relations.

- When the manifold is the momentum space, we characterised its geometry in terms of a four dimensional extension of the relative velocity (Lobachevsky) space, whose Riemann curvature is determined by the modified dispersion relation $F(p^2) = -m^2$.

- Thus, our work interconnects generalised uncertainty principle, momentum space geometry, and modified dispersion relations in a covariant setting.

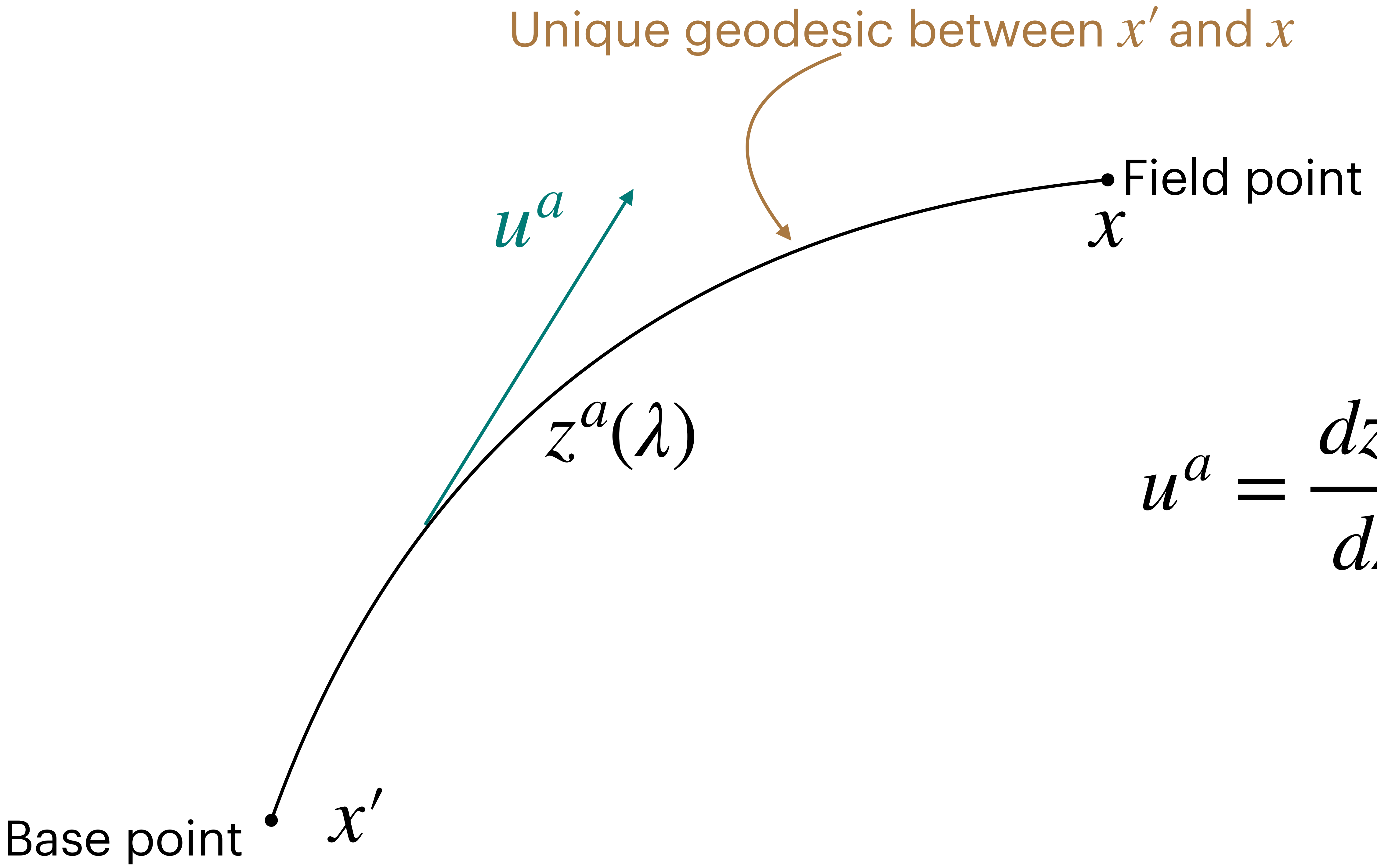
THANK YOU

Biscalar and Synge's world function

- Synge's world function is a scalar function of the base point x' and the field point x.

$$\sigma(x, x') = \frac{\lambda_1 - \lambda_0}{2} \int_{\lambda_0}^{\lambda_1} g_{ab} u^a u^b d\lambda$$

$$\sigma = \frac{(\Delta\lambda)^2}{2}\epsilon$$



$$u^a = \frac{dz^a}{d\lambda}$$

- The world function $\sigma(x, x')$ can be differentiated with respect to either argument ,

$$\sigma_a = \Delta\lambda u_a, \quad \sigma_{a'} = -\Delta\lambda u_a\,.$$

Synge's world function..

- In flat spacetime, the geodesic linking x to x′ is a straight line, and

$$\sigma = \frac{1}{2}\eta_{ab}(x - x')^a(x - x')^b$$

- σ_a behaves as a dual vector with respect to tensorial operations carried out at x, but as a scalar respect to operations carried out x'.

- The limiting behaviour of the bitensors $\sigma_{\dots}$ as x approaches x' is called coincidence limit of the bitensor.

$$[\sigma] = 0, \quad [\sigma_a] = 0$$

$$[\sigma_{ab}] = g_{ab} \qquad\qquad [\sigma_{abcd}] = -\frac{1}{3}(R_{acbd} + R_{bdac})$$

Geometry of momentum space

$$dl^2 = -dm^2 + \mu^2 dl_{\text{rel}}^2$$

- Where, $\mu = f(m)$, $dl_{\text{rel}}^2 = d\chi^2 + sinh^2\chi d\Omega^2$ is the Lobachevsky metric of the relativistic velocity space.

- The construction is motivated by the two-particle system with masses m_1, m_2 and four momenta p_1^i, p_2^j respectively and writing the energy of this system in it's center of momentum frame.

$$E_{\text{com}}^2 = \left(\mu_1 + \mu_2\right)^2 + l^2$$

$$l^2 = 2\mu_1\mu_2(\gamma_{\text{rel}} - 1)$$

$$\gamma_{\text{rel}} = -u_1 \,.\, u_2, \quad \mu = f(m)$$

Geometry of momentum space

$$p^2 = -m^2 \longrightarrow \mu = f(m)$$

$$\big\downarrow$$

$$p^2 = -\mu^2 \longrightarrow F(p^2) = -m^2$$

- We interpret l^2 as the measure of (squared) "three momentum distance" between the two particles

$$l^2 = E_{\mathrm{com}}^2 - \underbrace{[E_{\mathrm{com}}^2]_{v_{\mathrm{rel}}=0}}_{\text{"Rest energy" in the center-of- momentum frame}}$$

$$|\vec{p}\,|^2 = E^2 - m^2$$ For a point particle

Geometry of momentum space

- parameterizing u^a in terms of standard Lorentz transformations

$$u^a(\chi, \Omega^A) = (\cosh \chi) T^a + (\sinh \chi) N^a$$

- Where T^a, N^a are arbitrary unit timelike, spacelike vectors in the tangent space $T_{\mathscr{P}_0}(M)$, with $T^a N_a = 0$, and $\Omega^A = (\theta, \phi)$.

$$l^2 = 2\mu_1\mu_2 \left(\cosh \chi_1 \cosh \chi_2 - \sinh \chi_1 \sinh \chi_2 \cos \Omega - 1\right)$$

Geometry of momentum space

- l^2 is the measure of (squared) ‘‘three momentum distance" between the two particles.

$$g_{ab}^{3-\mathrm{mom}} = \lim_{\xi_2^a \to \xi_1^a} \frac{\partial^2}{\partial \xi_1^a \xi_2^b} \left(\frac{\ell^2}{2} \right) \xrightarrow[\xi^a = (\mu, \chi, \Omega^A)]{} \mu^2 dl_{\mathrm{rel}}^2 = \mu^2 (d\chi^2 + sinh^2 \chi d\Omega^2)$$

- This, therefore, gives a rigorous justification for our definition of distance measure.

- It correctly gives a locally Lorentz invariant measure of relative momentum on the space of three momenta.