References (at all levels)

→ Hartle (beginner) (physics oriented approach)
→ Shutz (")

→ Sean Caroll (intermediate) (mathematically oriented)

→ E. Poisson (") (evenly math + phys.) (assumes some familiarity)

→ R. Wald (advanced) (mathematically rigorous (at physics level))

→ Mathias blau (lecture notes) (Amazing wealth of info.)

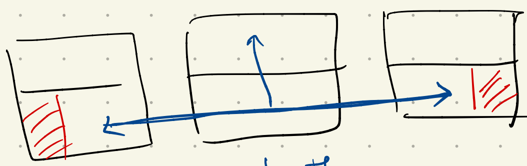
→ Landau Lifshitz } Classics.
→ MTW

→ Paddy's book - (intermediate) (very unique selection/organisation of topics)

→ Weinberg's Grav. & Cosm. - (")

→ very Reall's lecture notes (2016) (advanced intermediate).

Board Plan



only these boards should be sufficient.

- introduce.
- Ask about background.
- Ask questions

BIG PICTURE : GRAVITY = GEOMETRY (why?)

(and why should I learn fancy schmancy math?)

PHYSICAL THEORY

KINEMATICS

eg. $\dfrac{d^2\vec{x}}{dt^2} = \dfrac{\vec{F}}{m_I}$ valid for all theories

in components $\dfrac{d^2x^i}{dt^2} = \dfrac{F^i}{m_I}$

DYNAMICS

$\vec{F} = k\dfrac{q_1 q_2}{r^2}\hat{r}$

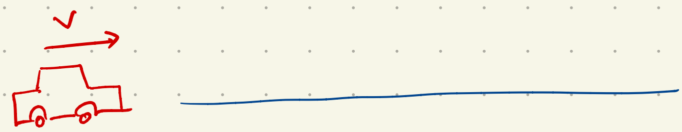
specific theory.

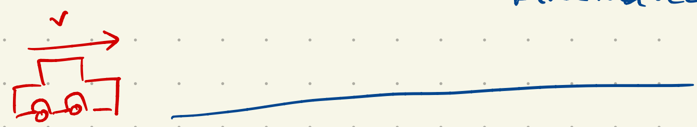
eg. Coulomb's law

let us forget history & think about this set-up for a second.

① Is our kinematic structure very general?

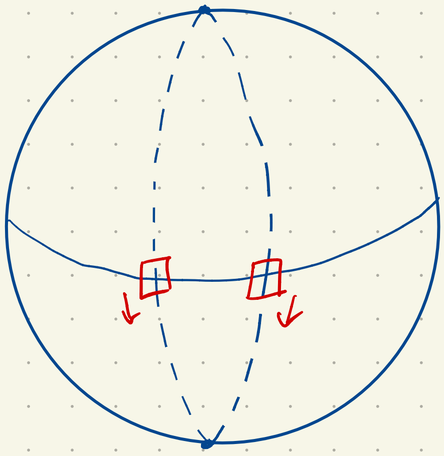
eg. ① Consider two cars with initial velocity $\vec{v}$ & no force acting on them. Also let them be parallel





kinematics will say they will never meet.

eg. ② What if they are on the surface of a sphere S^2 - let's say equator?



(Also about perceived force!)

They meet!

So already even at a Newtonian level we would like to generalise our kinematic structure.

→ this change is only at the level of space! No change in our understanding of time!

→ Already differential geometry is useful. (useful exercise, after this course, rederive Griffith's E.D cover page eqns.)

Back to semi-historical

$F = \frac{kq_1 q_2}{r^2}$ → only static situations.

↓ more dynamically Maxwell's eqtns.

$\partial_\mu F^{\mu\nu} = j^\nu$ $\left(\nabla \cdot E = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{c^2 \partial t} + \mu_0 \vec{j} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \cdot \vec{B} = 0 \right)$

Special relativity! ← speed of light universal!

(notion of time changed!)

Now we have notion of space-time!

Kinematic structure has undergone a dramatic change! Notion of simultaneity is not universal

Also $E = mc^2 \equiv$ mass - Energy equivalence!

More interesting geometric structure!

So far, so good.

Still why is Gravity = Geometry? Let's focus on dynamics.

Newton's laws $\vec{F} = -\frac{GMm}{r^2}\hat{r}$ (or $\nabla^2\phi = 4\pi G\rho$)

static, dynamics is not SR.

not a 4 vector

are incompatible with special relativity

(this was Einstein's motivation – purely theoretical)

For example, precession of the perihelion of the orbit of mercury is just a correction. Need not be geometrical!

But Gravity = Geometry gives ⟶ for free!

Gravity = Geometry aka Equivalence principle.

kinematics $\Leftrightarrow$ dynamics?

For simplicity, let us focus on uniform fields.

kinematical eqtn. $\vec{F} = m_I \vec{a}$

dynamical eqtn $\vec{F} = q\vec{E}$: Electrostatics.

$$\Rightarrow \vec{a} = \frac{q}{m_I}\vec{E}$$

dynamical eqtn. $\vec{F} = m_g \vec{g}$

$$\Rightarrow \vec{a} = \frac{m_g}{m_I}\vec{g}$$

What is the difference between the two?

$m_g = m_I$ (if it was proportional?)

Experimentally true.

Weak equivalence principle.

$\boxed{\rightarrow \text{Explain elevator expt. All cases.}}$

→ talk about light.

$\boxed{\rightarrow \text{Einstein equivalence principle.}}$

→ Bending of light.

→ Tidal forces X don't cover.

→ Gravitational Redshift.

talk about Lagrangian length & arc

$$\int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

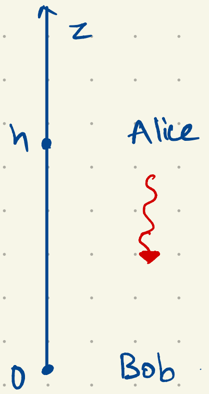
$$\int \sqrt{\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{d\vec{x}}{d\tau}\right)^2}$$

$$\int \sqrt{\left(\frac{dt}{d\tau}\right)^2\left(1 - \left(\frac{d\vec{x}}{dt}\right)^2\right)}$$

$$= \int d\tau \frac{dt}{d\tau}\sqrt{1-v^2}$$

$$\omega = \int dt \left(1 - \frac{v^2}{2} + \cdots\right)$$

Gravitational red-shift (weak field, non-relativistic)



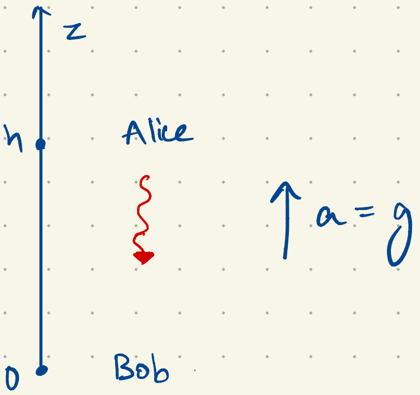
has a beacon that emits light signals in intervals of $\Delta\tau_A$

$\downarrow g$ uniform gravitational field.

what is $\Delta\tau_B$ (they have identical clocks to begin with)

(Pound-Rebka expt.)

Assume EEP holds,



$z_A = h + \frac{1}{2}gt^2$

$z_B = \frac{1}{2}gt^2$

$v_A = v_B = gt$

$\frac{v}{c} \ll 1$ (No S.R)

$\Rightarrow \frac{gt}{c} \ll 1$

Say Alice emits at $t = t_1$, the trajectory of that signal is $z(t) = z_A(t_1) - c(t - t_1)$

$$= h + \frac{1}{2}gt_1^2 - c(t - t_1)$$

let's say it meets Bob at T_1

$$\Rightarrow z_B(T_1) = h + \frac{1}{2}gt_1^2 - c(T_1 - t_1) = \frac{1}{2}gT_1^2 \quad -①$$

Alice emits a second signal at $t_2 = t_1 + \Delta\tau_A$

$$z(t) = z_A(t_1+\Delta\tau_A) - c(t - t_1 - \Delta\tau_A)$$

it meets Bob at $T_2 = T_1 + \Delta\tau_B$ (this defines $\Delta\tau_B$)

$$z_B(T_2) = z(T_2) = z_A(t_1+\Delta\tau_A) - c(t-t_1-\Delta\tau_A)$$

$$② \; - \; \boxed{\frac{1}{2}g(T_1+\Delta\tau_B)^2 = h + \frac{1}{2}g(t_1+\Delta\tau_A)^2 - c(T_1+\Delta\tau_B - t_1 - \Delta\tau_A)}$$

$$\frac{1}{2}gT_1^2 + gT_1\Delta\tau_B = h + \frac{1}{2}gt_1^2 + gt_1\Delta\tau_A - c(T_1-t_1) - c(\Delta\tau_B - \Delta\tau_A)$$

$-①$

$$\Rightarrow \quad \Delta\tau_B(gT_1 + c) = \Delta\tau_A(gt_1 + c)$$

$$\Delta\tau_B = \left(1+\frac{gt_1}{c}\right)\left(1+\frac{gT_1}{c}\right)^{-1}\Delta\tau_A$$

$$= \left(1+\frac{gt_1}{c}\right)\left(1-\frac{gT_1}{c}\right)\Delta\tau_A$$

$$\approx \left(1 + g\frac{(t_1-T_1)}{c}\right)\Delta\tau_A$$

$$(T_1 - t_1) = \frac{h}{c}$$

$$\Rightarrow \quad \Delta\tau_B \approx \left(1 - \frac{gh}{c^2}\right)\Delta\tau_A$$

Applying this to light with time period $\Delta\tau_A = \frac{\lambda_A}{c}$, $\Delta\tau_B = \frac{\lambda_B}{c}$

$$\lambda_B \approx \left(1 - \frac{gh}{c^2}\right)\lambda_A$$

(undergoes blue shift)

(redshift if Bob sends light to Alice)

$$\lambda_B \approx \left(1 + \frac{\Delta\phi}{c^2}\right)\lambda_A$$

(2 other types of blue shift / redshift)

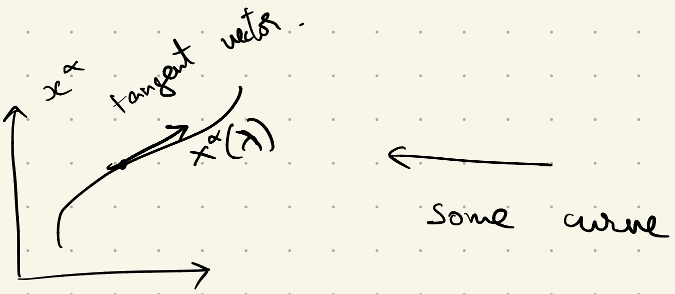
- → relativistic Doppler redshift
- → Cosmological redshift

OUTLINE

(-1) ~~Motivation & Big picture~~
(0) ~~Manifold.~~
(1) ~~Vectors / 1 forms. / Tensors.~~
(2) ~~Metric & inverse metric.~~
(3) ~~Different tangent spaces & connections.~~
(4) ~~Covariant derivative & parallel transport~~
(5) ~~Geodesics~~
(6) Lie derivative & Lie transport
(7) Killing vectors & Symmetries
— can be done after G.D.E
(8) ~~local flatness~~
(9) ~~Curvature~~
(10) Geodesic deviation
(11) Einstein's eqtns.
(12) Gravitational redshift
(13) ?? Schwarzschild / FRW ? — No time mostly.

Vectors, dual vectors & Tensors.

(informally explain Manifold)



Let's find out rate of change of a function $f(x^\alpha)$ along a curve $x^\alpha(\lambda)$

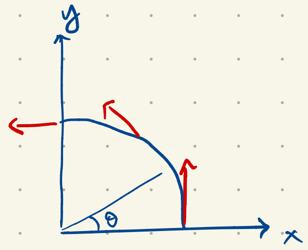
$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda} = \partial_\alpha f \, u^\alpha$$

2 objects

gradient (dual vector)

tangent vector to the curve

Example



eg.

$x = \cos\theta \qquad \theta \in [0, \pi/2]$

$y = \sin\theta$

$x^\alpha = (\cos\theta, \sin\theta)$

$\frac{dx^\alpha}{d\theta} = (-\sin\theta, \cos\theta)$

at $\theta = 0$, $(0,1)$

$\theta = \frac{\pi}{4}$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\theta = \pi/2$, $(-1, 0)$

under co-ordinate transformations,

$$x^\alpha \rightarrow y^{\alpha'}(x^\alpha)$$

$$\partial_{\alpha'} f = \frac{\partial x^\alpha}{\partial y^{\alpha'}} \frac{\partial f}{\partial x^\alpha} = \frac{\partial x^\alpha}{\partial y^{\alpha'}} \partial_\alpha f$$

$\|^{ly}$

$$u^{\alpha'} = \frac{dy^{\alpha'}}{d\lambda} = \frac{\partial y^{\alpha'}}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda} = \frac{\partial y^{\alpha'}}{\partial x^\alpha} u^\alpha$$

$\Rightarrow \frac{df}{d\lambda} \equiv$ invariant.

as it should be.

Vector : (V^α): any object V^α which transforms as
(contravariant)

$$V^{\alpha'} = \frac{\partial y^{\alpha'}}{\partial x^\alpha} V^\alpha$$ under co-ord. transf.

Co-vector / 1-form / dual vector (P_α) : any object P_α which transforms as
(covariant)

$$P_{\alpha'} = \frac{\partial x^\alpha}{\partial y^{\alpha'}} P_\alpha$$

<u>Contraction</u> $V^{\alpha} P_{\alpha} \equiv$ invariant & is therefore a scalar.

<u>Generalising</u>

<u>TENSOR</u> of type (m,n) : $T^{\alpha_1 \alpha_2 \cdots \alpha_m}{}_{\beta_1 \beta_2 \cdots \beta_n}$ (order of indices matter!)

such that $$T^{\alpha_1' \alpha_2' \cdots \alpha_m'}{}_{\beta_1' \beta_2' \cdots \beta_3'} = \frac{\partial y^{\alpha_1'}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\alpha_m'}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial y^{\beta_1'}} \cdots \frac{\partial x^{\beta_n}}{\partial y^{\beta_n'}} T^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n}$$

vector $\equiv (1,0)$ tensor

co-vector $\equiv (0,1)$ tensor.

A very special tensor of type $(0,2)$: <u>METRIC</u> $g_{\mu\nu}$ (Symmetric)

→ generalisation of the dot product.

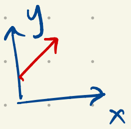
$$V \cdot W = g_{\mu\nu} V^{\mu} W^{\nu}$$

$$\|V\| = g_{\mu\nu} V^{\mu} V^{\nu}$$

invariant line element $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$

$(+---)$ or $(-+++)$ conventions

eg. Euclidean $\mathbb{R}^2$ in cartesian $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$V = (V^x, V^y) = (1,1)$; $\|V\| = 2$

at $x=0, y=1$

Polar co-ordinates $g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$r^2 = x^2 + y^2$

$\tan^{-1}\left(\frac{y}{x}\right) = \theta$

$$V^r = \frac{\partial r(x)}{\partial x} V^x + \frac{\partial r}{\partial y} V^y$$
$$= \frac{x}{r} + \frac{y}{r}$$
$$= 1$$

$$V^{\theta} = \frac{\partial \theta(x,y)}{\partial x} V^x + \frac{\partial \theta(x,y)}{\partial y} V^y$$

$$= -1$$

$$V^\mu \xrightarrow[g_{\mu\nu}]{} V_\mu = g_{\mu\nu} V^\nu$$

$$\|V\| = V^\mu V_\mu$$

$$V \cdot W = V_\mu W^\mu = V^\mu W_\mu$$

INVERSE METRIC $(g^{\mu\nu})$ (2,0) tensor

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$$: Kronecker delta (Identity matrix)

$$\tilde{V} \cdot \tilde{W} = V_\mu W_\nu g^{\mu\nu}$$

$$W_\mu \xrightarrow[g^{\mu\nu}]{} W^\mu = g^{\mu\nu} W_\nu$$

Explain the idea of a tangent plane (tgts. don't directly belong to the manifold)

- Tensors can be added, multiplied & contracted at a given point and is still a tensor.

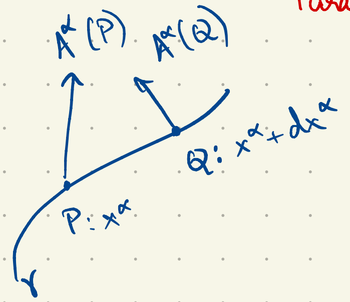
- A Tensor at pt. P & pt. Q cannot be combined in a tensorial way.

  eg: $A^\alpha(P) - A^\alpha(Q) = ??$

Derivatives ?

How do we compare Tensors a 2 different tangent spaces? We need some $\underbrace{\text{type of rule.}}_{\text{Parallel transport}}$

Consider a curve γ & a vector field A^α defined in its neighborhood.



2d plane eg:

$$\vec{V} = V^x \hat{x} + V^y \hat{y}$$
$$\vec{V} = V^i \hat{e}_i$$ cartesian
$$\partial_\alpha \vec{V} = (\partial_\alpha V^i) \hat{e}_i$$

(when $i = x, y$)

but $\vec{V} = V^r \hat{r} + V^\theta \hat{\theta}$
$= V^a \hat{f}_a$ polar co-ord.

Now $\hat{f}_a$ depends on (x,y) $(a = r, \theta)$

Read this after understanding Γ^i_{jk}

Give 2d polar example

$$dA^\alpha = A^\alpha(Q) - A^\alpha(P)$$
$$= A^\alpha(x^\beta + dx^\beta) - A^\alpha(x^\beta)$$
$$= \partial_\beta A^\alpha \, dx^\beta$$

is not tensorial.

$$\partial_\alpha \vec{V} = (\partial_\alpha V^a)\hat{f}_a + V^a(\partial_\alpha \hat{f}_a)$$

$$\partial_\alpha \hat{f}_a = \Gamma^b_{\alpha a} \hat{f}_b$$ (since $\partial_\alpha \hat{f}_a$ can also be expressed in same basis)

$$\Rightarrow \partial_\alpha \vec{V} = (\partial_\alpha V^a + \Gamma^a_{\alpha b} V^b)\hat{f}_a$$
$$= (\nabla_\alpha V^a)\hat{f}_a$$

$$\partial_{\beta'} A^{\alpha'} = \frac{\partial x^\beta}{\partial y^{\beta'}} \frac{\partial}{\partial x^\beta}\left(\frac{\partial y^{\alpha'}}{\partial x^\alpha} A^\alpha\right) = \underbrace{\frac{\partial x^\beta}{\partial y^{\beta'}} \frac{\partial y^{\alpha'}}{\partial x^\beta} \partial_\beta A^\alpha}_{\text{tensorial}} + \underbrace{\frac{\partial x^\beta}{\partial y^{\beta'}} \frac{\partial^2 y^{\alpha'}}{\partial x^\beta \partial x^\alpha}}_{\text{non-tensorial}} A^\alpha$$

We want to define

$$DA^\alpha = A^\alpha_T(P) - A^\alpha(P)$$

$\hookrightarrow$ transporting $A^\alpha(Q)$ to P

$$DA^\alpha = dA^\alpha + \delta A^\alpha$$

$$\delta A^\alpha = A^\alpha_T(P) - A^\alpha(Q)$$

How to define δA^α ?

demand it is linear in A^μ & dx^α

$$\delta A^\alpha = \underbrace{\Gamma^\alpha_{\mu\beta}}_{\text{connection}} A^\mu dx^\beta$$

Now $$\frac{DA^\alpha}{d\lambda} = \underbrace{\left(\partial_\beta A^\alpha + \Gamma^\alpha_{\mu\beta} A^\mu\right)}_{\nabla_\beta A^\mu} u^\beta$$

∇_β : covariant derivative (of a vector)

(some literature uses $A^\mu{}_{,\beta}$ for $\partial_\beta A^\mu$ & $A^\mu{}_{;\beta}$ for $\nabla_\beta A^\mu$)

Demanding that $\nabla_\beta A^\mu$ transforms covariantly, we get the transformation property of $\Gamma^\alpha_{\mu\beta}$

$$\Gamma^\alpha_{\mu\beta} A^\mu = \nabla_\beta A^\alpha - \partial_\beta A^\alpha \qquad \boxed{(\forall \; A^\mu)}$$

Exercise:

$$\boxed{\Gamma^{\alpha'}_{\mu'\beta'} = \frac{\partial y^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\beta}{\partial y^{\beta'}} \Gamma^\alpha_{\mu\beta} - \frac{\partial^2 y^{\alpha'}}{\partial x^\mu \partial x^\beta} \frac{\partial x^\beta}{\partial y^{\beta'}} \frac{\partial x^\mu}{\partial x^{\mu'}}}$$

Cov. derivative can be generalised to other types of tensors using Leibnitz rule.

$$d(A^\alpha p_\alpha) = D(A^\alpha p_\alpha) = (DA^\alpha) p_\alpha + A^\alpha (Dp_\alpha)$$

or

$$\partial_\beta (A^\alpha p_\alpha) = \nabla_\beta (A^\alpha p_\alpha)$$

$$(\partial_\beta A^\alpha) p_\alpha + A^\alpha \partial_\beta p_\alpha = (\nabla_\beta A^\alpha) p_\alpha + A^\alpha \nabla_\beta p_\alpha$$

$$\Rightarrow \nabla_\beta P_\alpha = \partial_\beta P_\alpha - \Gamma^\mu_{\alpha\beta} P_\mu$$

$$\nabla_\mu T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} = \partial_\mu T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} + \Gamma^{\alpha_1}_{\mu\sigma} T^{\sigma \alpha_2 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} + \Gamma^{\alpha_2}_{\mu\sigma} T^{\alpha_1 \sigma \alpha_3 \dots \alpha_m} \dots - \Gamma^\sigma_{\mu\beta_1} T^{\alpha_1 \dots \alpha_m}{}_{\sigma \beta_2 \dots \beta_n} - \dots$$

Until now the connection is completely arbitrary.

In vanilla G.R, $\underline{\underline{\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}}}$ & $\underline{\underline{\nabla_\alpha g_{\mu\nu} = 0}}$ (preserves dot. product)

Levi-Civita conn

$$\Rightarrow \Gamma^\alpha_{\mu\nu} = g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})$$: Christoffell symbol.

A tensor field is said to be parallel transported along a curve γ if its covariant derivative along the curve vanishes :
$$\frac{DT^{\alpha\ldots}{}_{\beta\ldots}}{d\lambda} = u^{\nu}\nabla_{\nu} T^{\alpha\ldots}{}_{\beta\ldots} = 0$$

eg: Sphere.

GEODESICS $\longleftarrow$ Recap what Tenoy says. maybe connect to parallel transport.

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}{}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0 \quad \text{(affinely parameterised)}$$

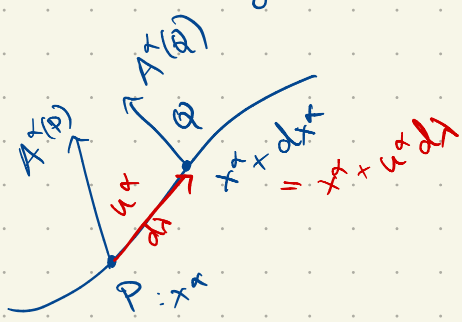
(if possible show $\epsilon = u^{\alpha}u_{\alpha}$ const. along affinely parameterised geodesic)

also $u^{\alpha}\nabla_{\alpha}u^{\beta} = 0$
(tangent vector is parallel transported along the geodesic)

Lie differentiation : (required for symmetries & Geodesic deviation eqtn.)

Covariant derivative required extra structure : $\Gamma^{\mu}{}_{\alpha\beta}$. $\longleftarrow$ path dependent but valid for every path.

Consider a curve γ $\equiv$ integral curve
tgt. vector $u^{\alpha} = \dfrac{dx^{\alpha}}{d\lambda}$ $\equiv$ vector field



Now $x'^{\alpha} = x^{\alpha} + u^{\alpha}d\lambda$
can be interpreted as a co-ordinate transformation

$$A^{\alpha'}(x') = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} A^{\beta}(x)$$
$$= (\delta^{\alpha}{}_{\beta} + \partial_{\beta}u^{\alpha}d\lambda)A^{\beta}(x)$$
$$= A^{\alpha}(x) + \partial_{\beta}u^{\alpha}A^{\beta}(x)\,d\lambda$$

i.e $A^{\alpha'}(Q) = A^{\alpha}(P) + \partial_{\beta}u^{\alpha}A^{\beta}(x)\,d\lambda$

But $A^\alpha(Q) = A^\alpha(x^\beta + u^\beta d\lambda)$

$$= A^\alpha(x^\beta) + u^\beta \partial_\beta A^\alpha \, d\lambda$$

$$= A^\alpha(P) + u^\beta \partial_\beta A^\alpha(P)\, d\lambda$$

$$\mathcal{L}_u A^\alpha(P) = \frac{A^\alpha(Q) - A^{\alpha'}(Q)}{d\lambda}$$

$$= u^\beta \partial_\beta A^\alpha - \partial_\beta u^\alpha A^\beta$$

Exercise: $\mathcal{L}_u A^\alpha$ is a tensor. Show this.

For scalars

$$\mathcal{L}_u \phi = u^\alpha \partial_\alpha \phi \qquad \text{(Ex. show this)}$$

For covectors

$$\mathcal{L}_u (v^\alpha w_\alpha) = u^\beta \partial_\beta (v^\alpha w_\alpha) = u^\beta \partial_\beta v^\alpha \, w_\alpha + u^\beta v^\alpha \partial_\beta w_\alpha$$

$$(\mathcal{L}_u v^\alpha) w_\alpha + v^\alpha (\mathcal{L}_u w_\alpha) = u^\beta \partial_\beta v^\alpha \, w_\alpha + u^\beta v^\alpha \partial_\beta w_\alpha$$

$$(\cancel{u^\beta \partial_\beta v^\alpha} - \partial_\beta u^\alpha v^\beta) w_\alpha + v^\alpha (\mathcal{L}_u w_\alpha) = (\cancel{u^\beta \partial_\beta v^\alpha}) w_\alpha + u^\beta v^\alpha \partial_\beta w_\alpha$$

$$v^\alpha (\mathcal{L}_u w_\alpha) = v^\alpha (u^\beta \partial_\beta w_\alpha + w_\beta \partial_\alpha u^\beta) \qquad \forall\, v^\alpha$$

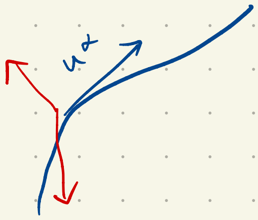
$$\Rightarrow \boxed{\mathcal{L}_u w_\alpha = u^\beta \partial_\beta w_\alpha + w_\beta \partial_\alpha u^\beta}$$

Ex. Generalise to arbitrary tensors.

Lie transport. $\mathcal{L}_u T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} = 0$

along a curve with tgt. vector u^α

If $\mathcal{L}_u T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} = 0$ then in the co-ordinate system $u^\alpha \overset{*}{=} \delta^\alpha_0$



$\Rightarrow \partial_\beta u^\alpha = 0$

"$\overset{*}{=}$" equality in a given co-ordinate system

$u^\gamma \partial_\gamma T^{\alpha_1 \dots}{}_{\beta_1 \dots \beta_n} = 0 \Leftarrow$

$\frac{\partial}{\partial x^0} T^{\alpha_1 \dots}{}_{\beta_1 \dots} = 0$. i.e $T^{\alpha_1 \dots}{}_{\beta_1 \dots}$ doesn't depend on x^0.

$\therefore$ Lie derivative is the natural operation to define symmetries.

Killing vectors

A symmetry of a spacetime is defined by the isometries of the metric.

~~eg~~ $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$

$$\left.\begin{aligned} \xi^\alpha_t &= (\partial_t)^\alpha = \frac{\partial x^\alpha}{\partial t} \\ \xi^\alpha_x &= (\partial_x)^\alpha = \frac{\partial x^\alpha}{\partial x} \\ \xi^\alpha_y &= (\partial_y)^\alpha = \frac{\partial x^\alpha}{\partial y} \\ \xi^\alpha_z &= (\partial_z)^\alpha = \frac{\partial x^\alpha}{\partial z} \end{aligned}\right\}$$

but rotations also exist.

doesn't seem manifest in these co-ordinates

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$\xi_\phi = \partial_\phi$ obvious. But ∂_x, ∂_y not obvious.

definition. $\rightarrow$ all ξ that satisfy Killing's eqtn.

$$\mathcal{L}_\xi g_{\mu\nu} = 0 = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

as before, in some co-ordinate system where $\xi^\nu \stackrel{*}{=} \delta^\nu_{(1)}$

then $\partial_{(1)} g_{\mu\nu} = 0$.

they help define conserved quantities for us. as well.

example $= u^\alpha \xi_\alpha$

$$\frac{d}{d\lambda}(u^\alpha \xi_\alpha) = u^\beta \nabla_\beta (\xi_\alpha u^\alpha)$$

$$= u^\alpha u^\beta \nabla_\beta \xi_\alpha + \xi_\alpha \cancel{u^\beta \nabla_\beta u^\alpha}^{\,0}$$

$$= \cancel{u^\alpha u^\beta \nabla_{(\beta} \xi_{\alpha)}} = 0$$

$$\frac{d\mathcal{E}}{d\lambda} = \frac{d(u^\alpha u_\alpha)}{d\lambda}$$

$$= (u^\beta \nabla_\beta u^\alpha) u_\alpha$$

$$+ u_\alpha u^\beta \nabla_\beta u$$

eg: $\xi^\alpha_{(t)} = \frac{\partial x^\alpha}{\partial t}$ $\quad E = m u^\alpha \xi^\alpha_{(t)}$

[Mention maximal symmetry]

Local flatness

The EEP says one can get rid of gravity by going to a freely falling frame. How do we implement this mathematically?

Ask & freestyle the discussion.

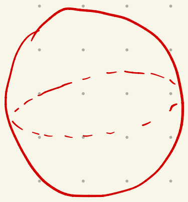
$$x'^\alpha = A^\alpha{}_\beta x^\beta + \mathcal{O}(x^2)$$

$$\Rightarrow g_{\alpha'\beta'} = A^\alpha{}_{\alpha'} A^\beta{}_{\beta'} g_{\alpha\beta}$$

$$\underset{\eta_{\alpha\beta}}{\underset{\Vert}{g_{\alpha'\beta'}}}$$

CURVATURE

What the hell is curvature? (think of 2d for the moment)

 – curved?

 – curved?

important to understand extrinsic & intrinsic curvature. How can an ant on S^2 understand ~~or~~ figure out curvature of the sphere? ~~eg~~ Area of circle in 2d or sum of angles in Δ

More generally, (tgt. space closed loop) ← explain conceptually. no computation

Define $R^{\mu}{}_{\nu\rho\sigma} A^{\nu} = [\nabla_\rho, \nabla_\sigma] A^{\nu} \quad \forall\ A^{\nu}$ (convention)

$$= \nabla_\rho \nabla_\sigma A^{\nu} - \nabla_\sigma \nabla_\rho A^{\nu}$$

$$\Rightarrow R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\rho\nu}$$

Properties

- $R_{[\mu\nu][\rho\sigma]}$ (3 properties)
- $R_{\mu[\nu\rho\sigma]} = 0$
- Bianchi identity

$$\nabla_\alpha R_{\mu\nu\rho\sigma} + \nabla_\sigma R_{\mu\nu\alpha\rho} + \nabla_\rho R_{\mu\nu\sigma\alpha} = 0$$

Contractions

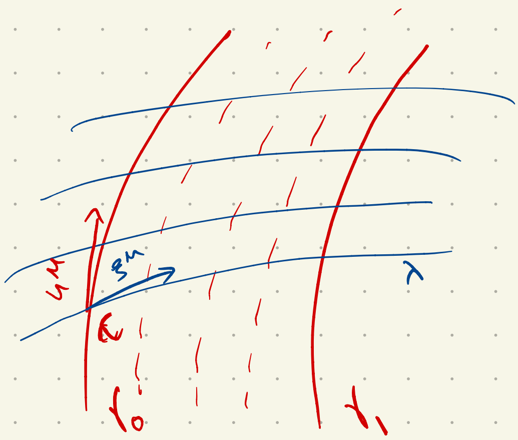
$$\left.\begin{aligned} R_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \\ R = g^{\mu\nu} R_{\mu\nu} &= R^{\alpha}{}_{\alpha} \end{aligned}\right\}$$

Bianchi $\Rightarrow$ $\nabla_\mu R^{\mu}{}_{\nu} = \frac{1}{2} \nabla_\nu R$

$$\nabla^\mu R_{\mu\nu} - \frac{1}{2} \nabla^\mu g_{\mu\nu} R = 0$$

$$\nabla^\mu \underbrace{\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right)}_{G_{\mu\nu}} = 0$$

<u>Geodesic</u> <u>deviation</u> (what is Riemann?)



Consider a geodesic congruence. $x^\mu(\tau, \lambda)$

$$x^\mu(\tau, 0) = \gamma_0$$

$$x^\mu(\tau, 1) = \gamma_1$$

$$\frac{\partial x^\mu}{\partial \tau} = u^\mu$$

$$\frac{\partial x^\mu}{\partial \lambda} = \xi^\mu$$

Now $\dfrac{\partial u^\mu}{\partial \lambda} = \dfrac{\partial \xi^\mu}{\partial \tau}$ (partial derivatives commute)

$$\Rightarrow \quad \xi^\alpha \nabla_\alpha u^\mu = u^\alpha \nabla_\alpha \xi^\mu$$

Acceleration of the separation vector,

$$\frac{D^2 \xi^\mu}{D\tau^2} = u^\alpha \nabla_\alpha \left(u^\beta \nabla_\beta \xi^\mu\right) = u^\alpha \nabla_\alpha \left(\xi^\beta \nabla_\beta u^\mu\right)$$

$$= u^\alpha (\nabla_\alpha \xi^\beta)(\nabla_\beta u^\mu) + u^\alpha \xi^\beta \nabla_\alpha \nabla_\beta u^\mu$$

$$= \xi^\alpha \nabla_\alpha u^\beta \nabla_\beta u^\mu + u^\alpha \xi^\beta [\nabla_\alpha, \nabla_\beta] u^\mu + u^\alpha \xi^\beta \nabla_\beta \nabla_\alpha u^\mu$$

$$= u^\alpha \xi^\beta R^\mu{}_{\lambda\alpha\beta} u^\lambda + \xi^\alpha \underbrace{\left(\nabla_\alpha u^\beta \nabla_\beta u^\mu + u^\beta \nabla_\alpha \nabla_\beta u^\mu\right)}_{0}$$

$$\frac{D^2 \xi^\mu}{D\tau^2} = u^\alpha u^\lambda \xi^\beta R^\mu{}_{\lambda\alpha\beta}$$

Einstein's eqtns

$$\nabla^2 \phi = 4\pi G \rho$$

in weak field limit of geodesic eqth., we saw $\phi \sim h_{00}$

$\Rightarrow$ generalisation of $\phi \rightarrow g_{\mu\nu}$

this is mass density.

$\downarrow$ tensor generalisation

$T_{\mu\nu}$ stress-energy/energy momentum tensor

$\nabla_\mu T^\mu{}_\nu = 0$ (generalisation of continuity eqtn.)

$$[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}$$

$(0,2)$ symmetric $\propto$ $(0,2)$ symmetric

$$\nabla^2 g_{\mu\nu} = 0$$

$R_{\mu\nu} = k T_{\mu\nu}$ (Einstein's first guess)

but $\nabla_\mu R^\mu{}_\nu = \frac{1}{2} \nabla_\nu R$

but $R = kT$

so if we want $\nabla_\mu T^\mu{}_\nu = 0 \Rightarrow \nabla_\mu R^\mu{}_\nu = 0$

$\Rightarrow \nabla_\nu R = 0 \Rightarrow \nabla_\nu T = 0$

$\Rightarrow \partial_\nu T = 0$

$\Rightarrow$ T is constant everywhere !?

for a mass distribution, $T \neq 0$ near it, but far away $T = 0$, so this cannot be the case.

But $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is perfect

$\nabla_\mu G^\mu{}_\nu = 0$ from Bianchi identity.

$$\boxed{\therefore \; G_{\mu\nu} = \kappa T_{\mu\nu} \quad : \text{Einstein's field eq}^{\text{tns}}}$$

In fact, conservation of energy-momentum is a consequence of the Bianchi identity!

→ a consequence of geometry!

κ is fixed by weak-field consistency with Newtonian gravity. $\kappa = \frac{8\pi G}{c^4}$

In absence of matter, $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$

$\Rightarrow R_{\mu\nu} = 0$

$G_{\mu\nu} = \kappa T_{\mu\nu}$ ≡ seems simple no?

HELL NO!

10 - coupled highly non-linear partial differential eq$^{\text{tns}}$. for the metric

WE CANNOT SOLVE THIS IN GENERAL!

But with assumptions of a high degree of symmetry, some solutions are known.

Stars & Black holes

~~eg.~~ Schwarzschild or $\underbrace{\text{Friedman - Robertson - Walker - Lemaitre}}_{\text{Cosmology}}$

Special relativity & other physics

or Minkowski ☺

Also Gravitational wave solutions! → GW's