November 25, 2022

Mini School on Gravitation and Cosmology

Lecture - I

Dhiraj Kumar Hazra, IMSc, Chennai

Resource: Dhiraj Kumar Hazra / Codes in Cosmology · GitLab

The Institute of Mathematical Sciences

Theory

The Institute of Mathematical Sciences

Concept that have been covered

Metric

Homogeneity and isotropy

Ages

Distances

Ingredients

Covered by Prof. Sethi

I will begin with the numerical implementation of these concepts

Friedmann Equations

Equating the *00* and *ij* components of Einstein equation:

$$-\frac{2K}{a^2} - \frac{2\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} = -4\pi G(\rho - p)$$

$$\frac{3\ddot{a}}{a} = -4\pi G(3p + \rho) \ .$$

Combining we get

$$\dot{a}^2 + K = \frac{8\pi G \rho a^2}{3} \qquad \dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p)$$

The H_0 and the critical density

In the main Friedmann equation, we define:

$$\dot{a}(t)/a(t) = H(t)$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right).$$

Density parameters:

Assuming we have 3 types of components (matter, radiation and dark energy)

Define $\rho_c = \frac{3H^2}{8\pi G},$ critical density

* Note that there will be factors of c in some equations and c=1 in some others

Densities

Define the density parameters

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}.$$

Usually we talk about Ω_{today} for the densities of the components today

The definition also fixes the total density to 1

$$\frac{H^2}{H_0^2} = \Omega_{0,R}a^{-4} + \Omega_{0,M}a^{-3} + \Omega_{0,k}a^{-2} + \Omega_{0,\Lambda}.$$

This is going to be an extremely useful relation. Most of the cosmology is built around integrating this function

Densities

The curvature density is defined as: $\Omega_K \equiv -\frac{K}{a_0^2 H_0^2}$

We need to be careful about the dimensions of K

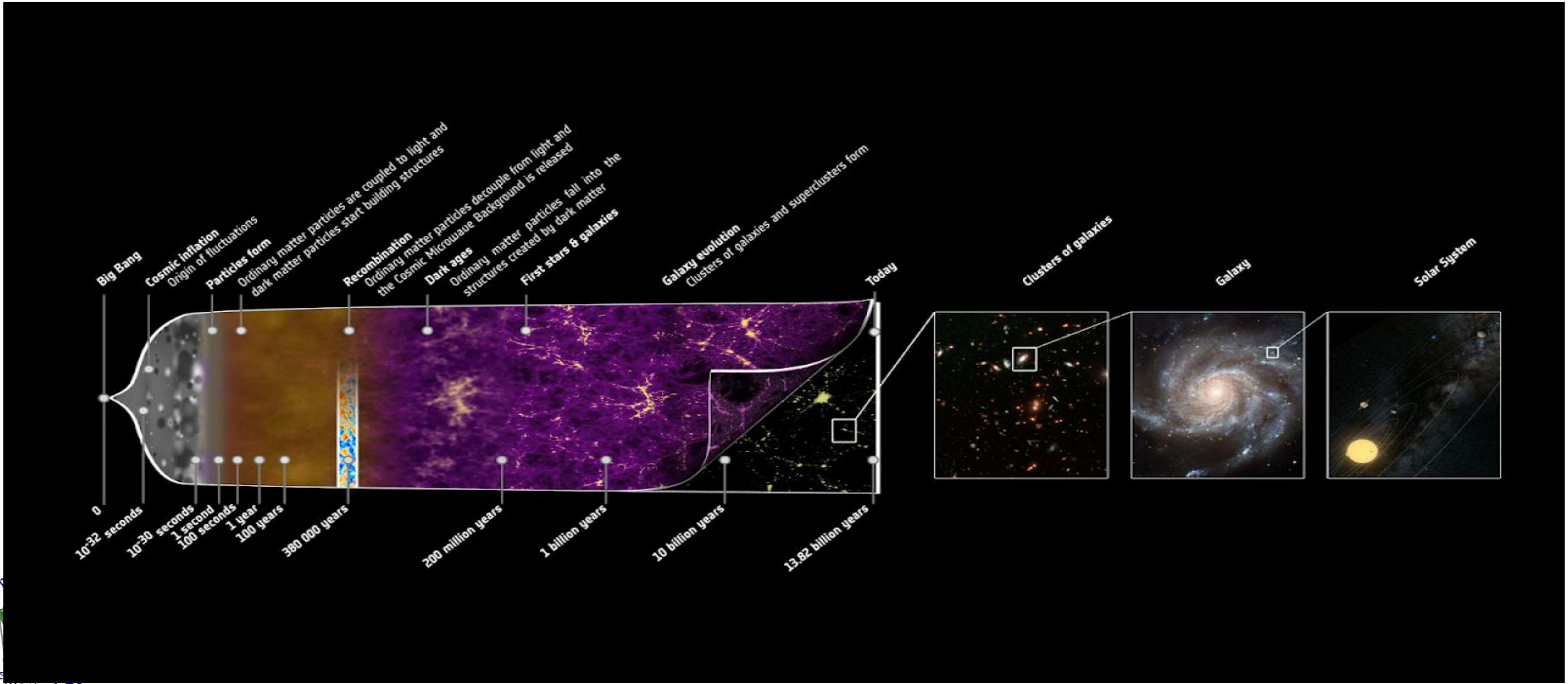
But note that we can keep density as free parameter when we talk about open/closed Universe

Absolute densities can be obtained easily from here:

$$\rho_{V0} \equiv \frac{3H_0^2\Omega_\Lambda}{8\pi G}\,, \quad \rho_{M0} \equiv \frac{3H_0^2\Omega_M}{8\pi G}\,, \quad \rho_{R0} \equiv \frac{3H_0^2\Omega_R}{8\pi G}$$

Evolution of the Universe

[(Image credit: Satellite: Planck, Copyright: ESA - C. Carreau)]



Age of the Universe

We can integrate this equation over the scale factor to get the age of the Universe

$$\frac{H^2}{H_0^2} = \Omega_{0,R}a^{-4} + \Omega_{0,M}a^{-3} + \Omega_{0,k}a^{-2} + \Omega_{0,\Lambda}.$$

Baseline model:

Flat FRW Universe with Einstein Gravity (cosmological constant as dark energy, *w=-1*), dark matter with only gravitational interaction. Primordial power spectrum is power law defined just by an amplitude and a tilt.

Age of the Universe

```
def dt_da_LCDM(a,Om_m,Om_l,Om_r,H0):
    Om_k=1-Om_m-Om_l-Om_r
    da_dt=H0*np.sqrt((Om_m*a)+Om_r+(Om_k*a**2)+Om_l*a**4)
    dt_da=a/da_dt
    return dt_da
```

```
Age= integrate.romberg(dt_da_LCDM_integrand,0,1
                        ,rtol=1e-3)*const.MPc/const.Km/const.Gyr
print("Age of the Universe today is:", f'{Age:.2f}','Gigayears')
```

```
Age of the Universe today is: 13.79 Gigayears
```

The Institute of Mathematical Sciences

Redshifts

The expansion redshifts the travelling lights

$$\frac{\lambda_0}{\lambda_e} = 1 + z = \frac{a(t_0)}{a(t_e)}$$

Plot the age of the Universe as a function of redshift → Assignment

Exchanging variables a → z, we can rewrite the earlier equations

Distances

We can similarly write

$$dt = \frac{dx}{H_0 x \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}$$

$$= \frac{-dz}{H_0 (1+z) \sqrt{\Omega_\Lambda + \Omega_K (1+z)^2 + \Omega_M (1+z)^3 + \Omega_R (1+z)^4}}$$

r(z) = $\int_{t(z)}^{t_0} \frac{dt}{a(t)}$ but this holds true only for flat cases

Distances in curved Universe

It becomes a function S(z) [Weinberg - Cosmology],

$$r(z) = S\left[\int_{t(z)}^{t_0} \frac{dt}{a(t)}\right]$$

$$= S\left[\frac{1}{a_0 H_0}\int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}\right]$$

$$S[y] \equiv \begin{cases} \sin y & K = +1 \\ y & K = 0 \\ \sinh y & K = -1 \end{cases}$$

Proper distance

Distance between two simultaneously occurring events (dt=0, dϴ = dφ =0)

$$s(t) = \int^{s} ds' = a(t) \int_0^r \frac{dr'}{(1 - Kr'^2)^{1/2}}$$

Proper distance to an object is

different than the coordinate distance

if the Universe is not flat.

$$s(t) = a(t) \cdot \begin{cases} \frac{1}{\sqrt{k}} \sin^{-1}(r\sqrt{k}) & \text{for } k > 0 \\ r & \text{for } k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh^{-1}(r\sqrt{|k|}) & \text{for } k < 0 \end{cases}$$

The Institute of Mathematical Sciences

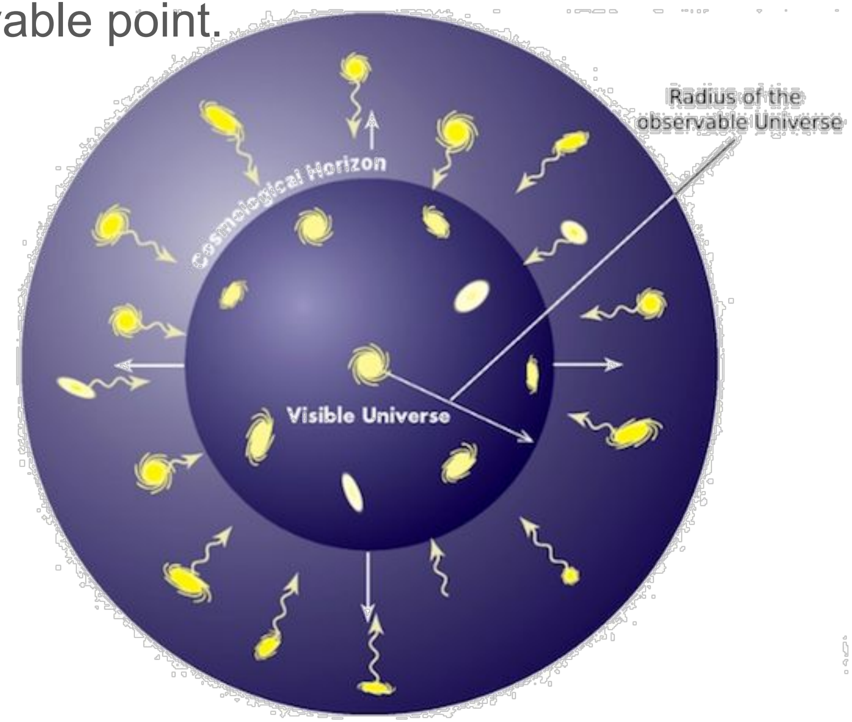
Horizon

It is the proper distance to the farthest observable point.

As the Universe expands, the size that we observe always increases.

Therefore we start to observe more distant objects

What is the horizon today ?



Horizon

for a flat Universe $s(t) = a(t)r$, using:

$$\int \frac{cdt}{a(t)} = \int \frac{dr'}{(1 - Kr'^2)^{1/2}}$$

$$d\eta = \int_0^a \frac{cda'}{a'^2 H(a)}$$

For curved Universe, this relation will change to:

$$r_{Horizon} = f(\int_0^t \frac{cdt}{a(t)})$$

,

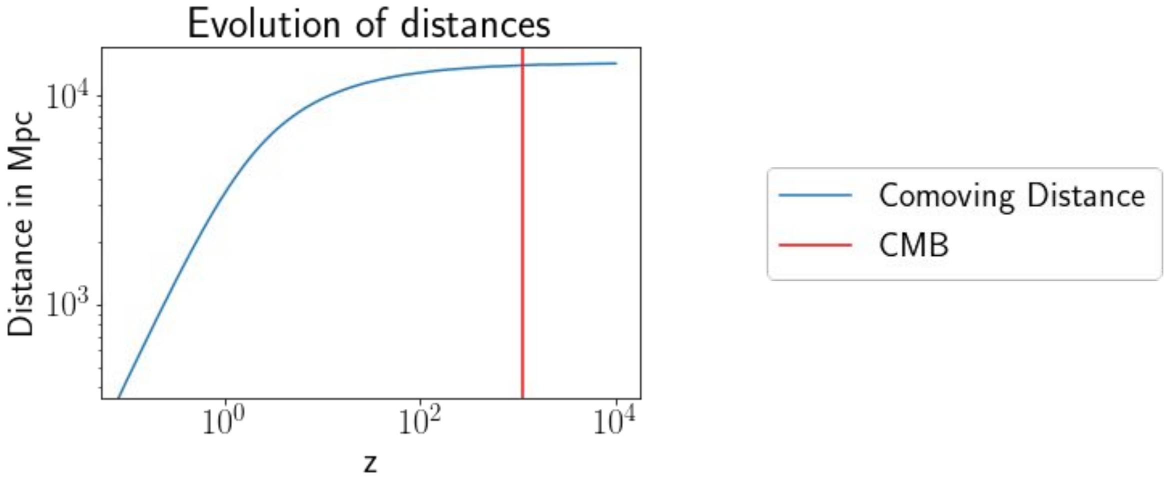
where this function, for $K = 1$, $f(x) = \sin(x)$; $K = 0$, $f(x) = x$ and $K = -1$, $f(x) = \sinh(x)$.

Comoving horizon is just without the scale factor. *Today* comoving and proper horizon are the same.

Horizon today is 14.26 Gpc = 46.5 billion light years

Comoving distance

It is the distance between objects in a space which doesn't expand. It would be constant had the Universe not been expanding (ref ***ipynb***)

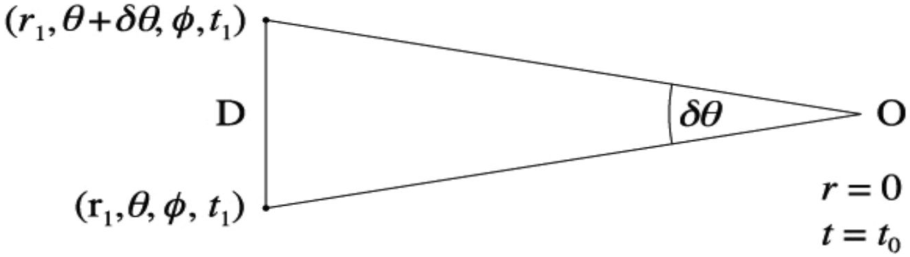


The slope mainly changes after the dark energy dominated era.

Angular diameter distance

Size of source is D

At radial distance r_1 and time t_1



Proper distance between the two ends $D=a(t_1)r_1\delta\theta$

Angular diameter distance is defined by:

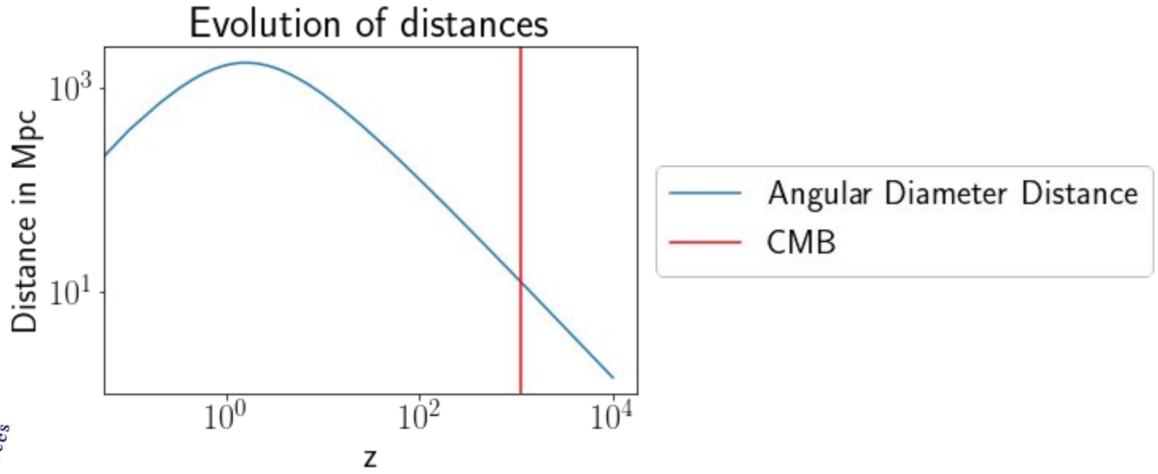
$d_A=D/\delta\theta \Rightarrow a(t_1)\, r_1 \Rightarrow r_1/(1+z_1)$

Angular diameter distance

$$d_A(z) = \frac{c}{\sqrt{|\Omega_{\mathrm{k},0}|}H_0(1+z)} \cdot S_k\left(H_0\sqrt{|\Omega_{\mathrm{k},0}|}\int_0^z \frac{dz}{H(z)}\right)$$

$$S_k(x) = \begin{cases} \sin(x) & \text{for } k > 0 \\ x & \text{for } k = 0 \\ \sinh(x) & \text{for } k < 0 \end{cases}$$

$d_A(z)$ is not a strictly increasing function of z (ref ***ipynb***)



Angular diameter distance

Objects at high redshifts appear larger than they are

The angular diameter distance have a maximum at z_{peak}

Therefore $\delta\theta = D/d_A$ leads to a minimum $\delta\theta$ at the z_{peak}

For the baseline best-fit cosmology, $z_{peak} = 1.5$. After this redshift objects appear bigger than their proper size

- Calculate z_{peak} for curved Universe (for both types)
- Calculate z_{peak} for different matter densities

Luminosity distance

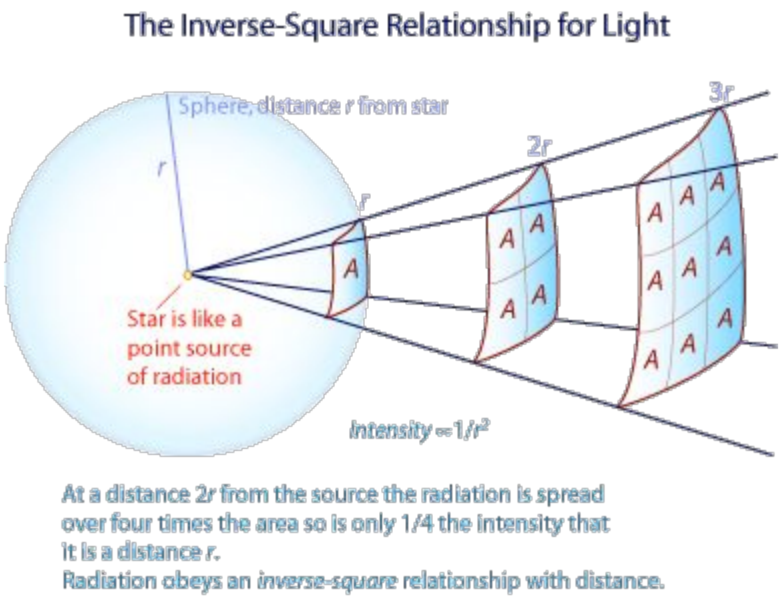
We need a distance that increases with time

In particular the distance that follows:

Flux = $L/(4\pi r^2)$

Radiation covers a greater surface area as they move away from the stars

Therefore intensity per unit area decreases as $1/r^2$



https://www.atnf.csiro.au/outreach/education/senior/cosmicengine/stars_luminosity.html

Luminosity distance

$d_L(z)=d_C(z)/a(z)$ satisfies these properties

This measure is the crucial input in obtaining constraints from SN data (lecture 3)

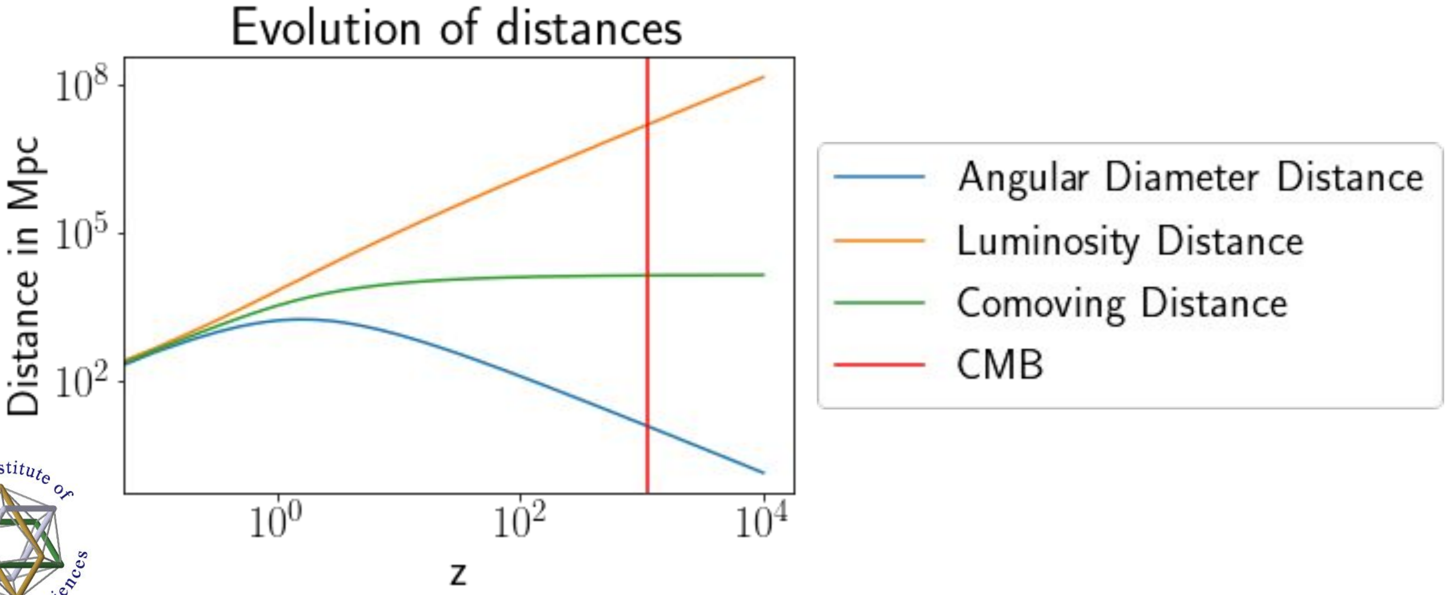
For a flat Universe, the luminosity distance will be (ref. Weinberg)

$$d_L(z) = a_0 r_1 (1+z) = \frac{1+z}{H_0} \int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4}}}$$

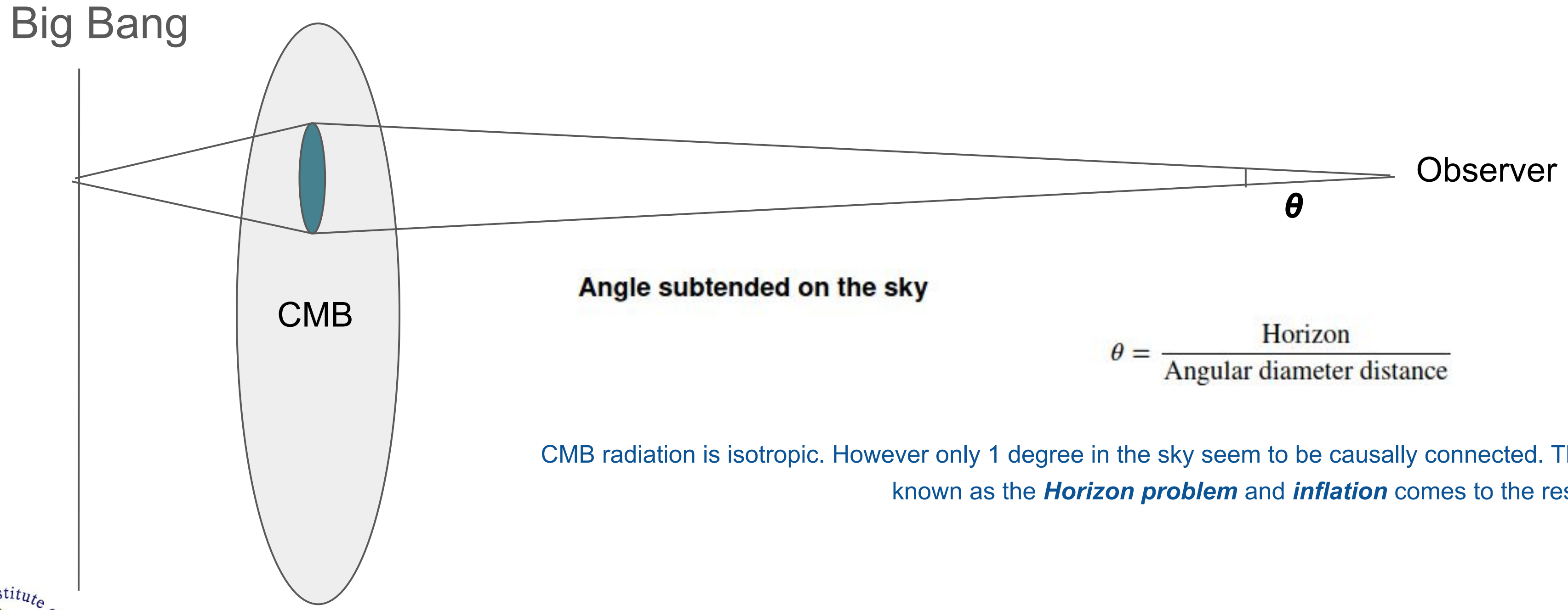
Relation between distances

$$d_L(z) = d_C(z)/a(z) = d_A(z)/a^2(z)$$

How distances change over redshift/time



Calculate angles (horizon problem)



Angle subtended on the sky

$$\theta = \frac{\text{Horizon}}{\text{Angular diameter distance}}$$

CMB radiation is isotropic. However only 1 degree in the sky seem to be causally connected. This is known as the ***Horizon problem*** and ***inflation*** comes to the rescue.

How to get constraints from observations?

Supernovae

Explosion of stars in binary systems

They have a peak brightness which is uniform

Can be used as standard candles

Their apparent brightness gives the information about the expansion of the Universe

The Institute of Mathematical Sciences

Distance modulus

Apparent magnitude - absolute magnitude = m - M = 5 log (d_L^{parsec}) -5

m-M=5log(d_L(Mpc))+25

Use luminosity distance and fit the Supernovae data

http://supernova.lbl.gov/Union/

580 SNe

Download the data and plot with errors http://supernova.lbl.gov/Union/

Download the error covariance matrix

The Institute of Mathematical Sciences

Absolute magnitude

THE ASTROPHYSICAL JOURNAL, 826:56 (31pp), 2016 July 20 RIESS ET AL.

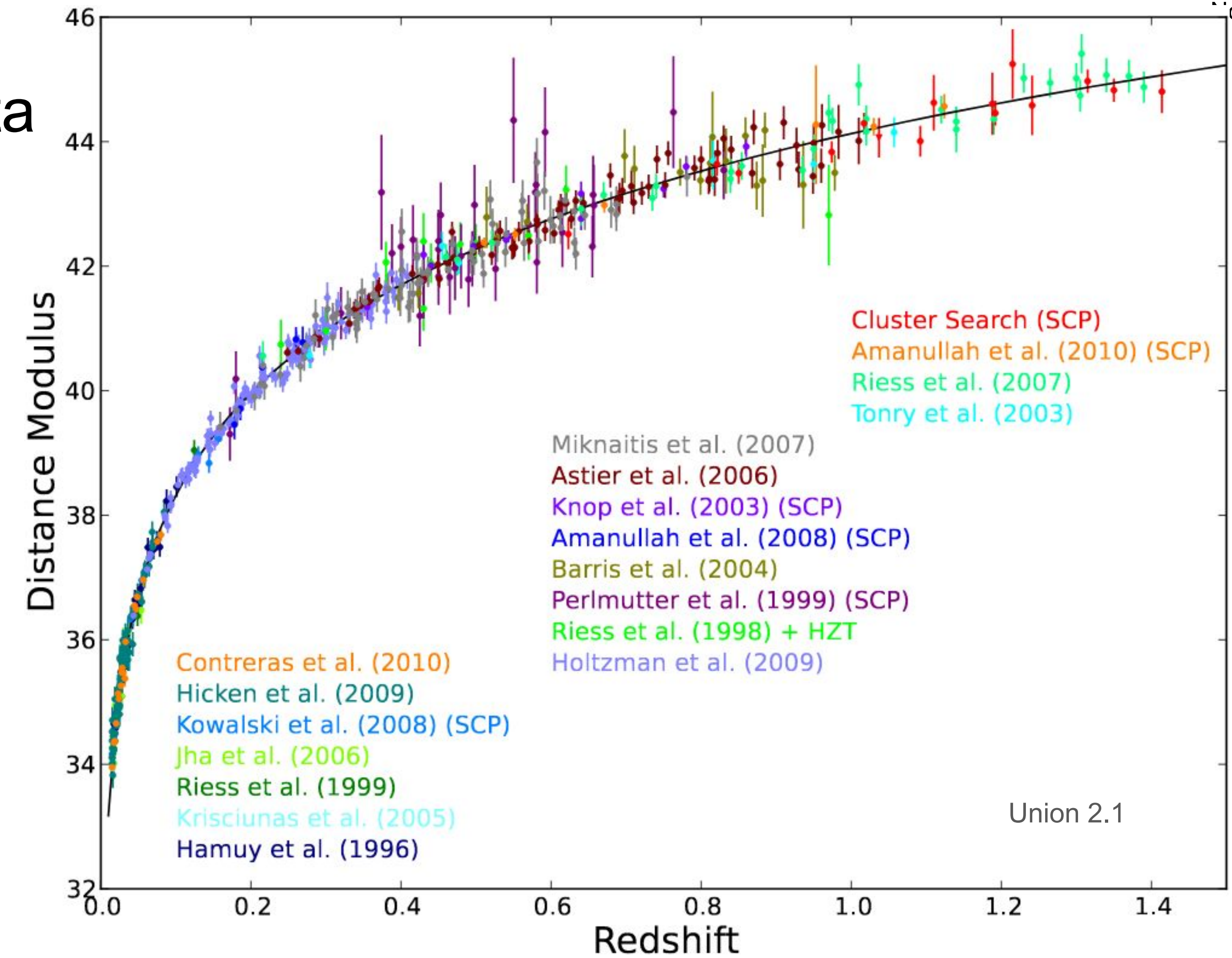
Table 5
Approximations for Distance Parameters

Host	SN	$m_{B,i}^{0} + 5a_B$	σ[a]	μ_{Ceph}[b] (mag)	σ	$M_{B,i}^{0}$	σ
M101	2011fe	13.310	0.117	29.135	0.045	−19.389	0.125
N1015	2009ig	17.015	0.123	32.497	0.081	−19.047	0.147
N1309	2002fk	16.756	0.116	32.523	0.055	−19.331	0.128
N1365	2012fr	15.482	0.125	31.307	0.057	−19.390	0.137
N1448	2001el	15.765	0.116	31.311	0.045	−19.111	0.125
N2442	2015F	15.840	0.142	31.511	0.053	−19.236	0.152
N3021	1995al	16.527	0.117	32.498	0.090	−19.535	0.147
N3370	1994ae	16.476	0.115	32.072	0.049	−19.161	0.125
N3447	2012ht	16.265	0.124	31.908	0.043	−19.207	0.131
N3972	2011by	16.048	0.116	31.587	0.070	−19.103	0.136
N3982	1998aq	15.795	0.115	31.737	0.069	−19.507	0.134
N4038	2007sr	15.797	0.114	31.290	0.112	−19.058	0.160
N4424	2012cg	15.110	0.109	31.080	0.292	−19.534	0.311
N4536	1981B	15.177	0.124	30.906	0.053	−19.293	0.135
N4639	1990N	15.983	0.115	31.532	0.071	−19.113	0.135
N5584	2007af	16.265	0.115	31.786	0.046	−19.085	0.124
N5917	2005cf	16.572	0.115	32.263	0.102	−19.255	0.154
N7250	2013dy	15.867	0.115	31.499	0.078	−19.196	0.139
U9391	2003du	17.034	0.114	32.919	0.063	−19.449	0.130

Notes.
[a] For SALT-II, 0.1 mag added in quadrature to fitting error.
[b] Approximate, SN-independent Cepheid-based distances as described at the end of Section 3.

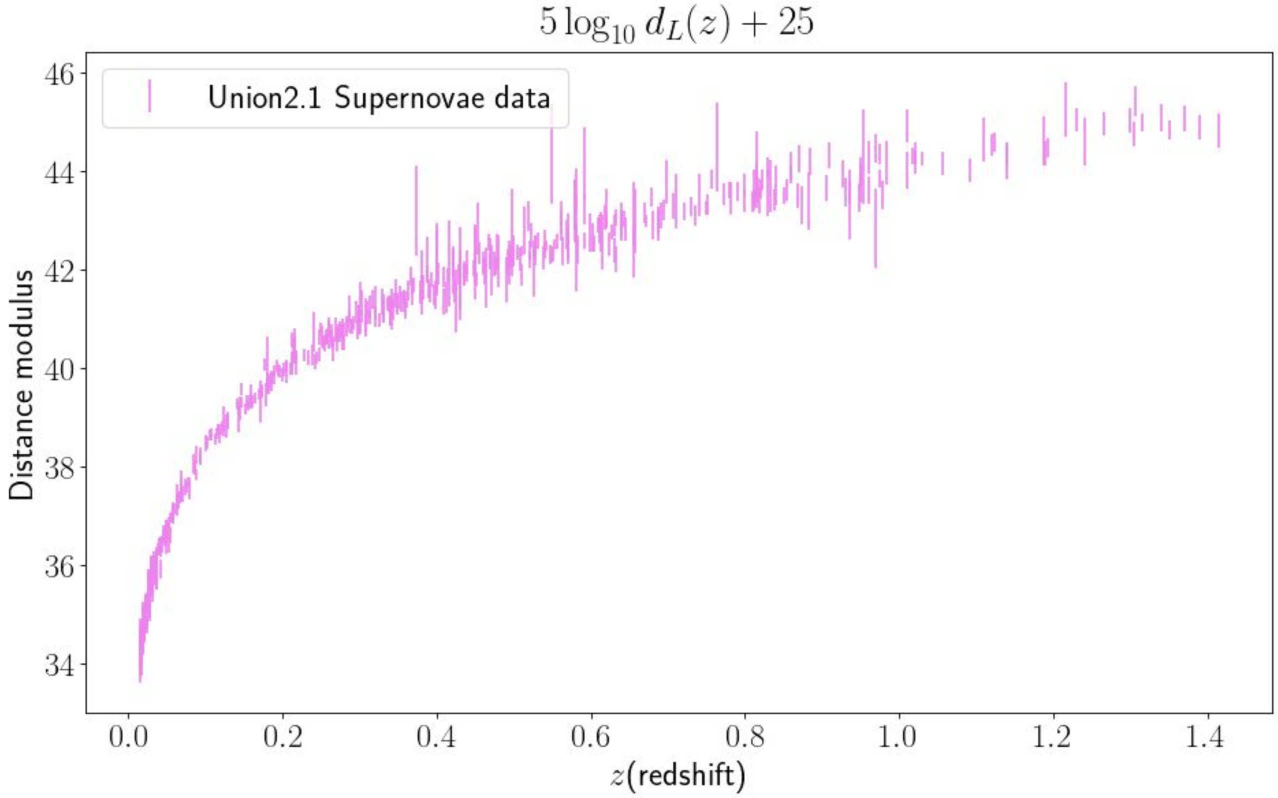
Absolute Magnitude ~ 19-19.5

The data



Distance Modulus
Redshift
46
44
42
40
38
36
34
32
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Cluster Search (SCP)
Amanullah et al. (2010) (SCP)
Riess et al. (2007)
Tonry et al. (2003)
Miknaitis et al. (2007)
Astier et al. (2006)
Knop et al. (2003) (SCP)
Amanullah et al. (2008) (SCP)
Barris et al. (2004)
Perlmutter et al. (1999) (SCP)
Riess et al. (1998) + HZT
Holtzman et al. (2009)
Contreras et al. (2010)
Hicken et al. (2009)
Kowalski et al. (2008) (SCP)
Jha et al. (2006)
Riess et al. (1999)
Krisciunas et al. (2005)
Hamuy et al. (1996)
Union 2.1

Plot the data



Least square fit

Given N data points $D(i)$ and errors $\sigma(i)$, We estimate our theory $T(i)$ for a given set of parameters Θ

.

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{D(i) - T(i)}{\sigma_i} \right)^2$$

Given error covariance, it can be written as:

$$\chi^2 = \sum_{i=1}^{N} [D(i) - T(i)]^T COV^{-1} [D(i) - T(i)]$$

Likelihood

From the least square, the likelihood (L) can be obtained as ***-2 In L =*** $\boldsymbol{\chi^2}$

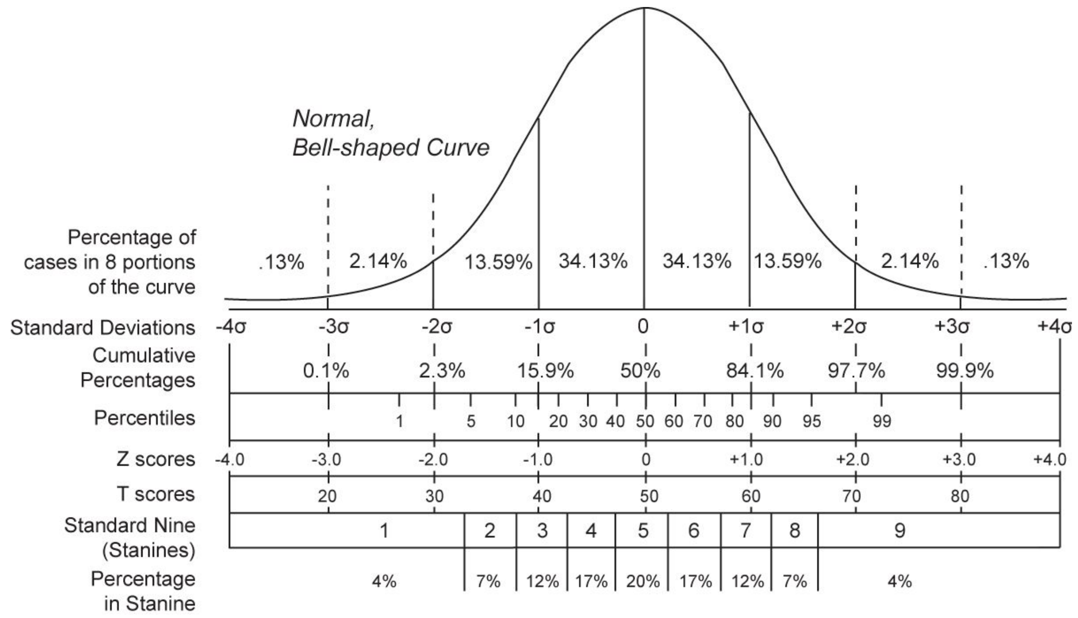
For smaller number of parameters one can simply do a grid search and plot the likelihood maps

Highest likelihood or least $\boldsymbol{\chi^2}$ represents the best fit

The probability density decreases as we go away from the best fit*

**** For Gaussian posteriors***

Confidence intervals



source : wikipedia

We need to get the confidence intervals for the cosmological parameters

In this example of constraints obtained using Type Ia SNe data $\boldsymbol{\Omega}_{\mathbf{m}}$ is the relevant parameter

H_0 is degenerate with the absolute magnitude. Therefore H_0 constraint from the SNe Union data does not represent the actual constraint

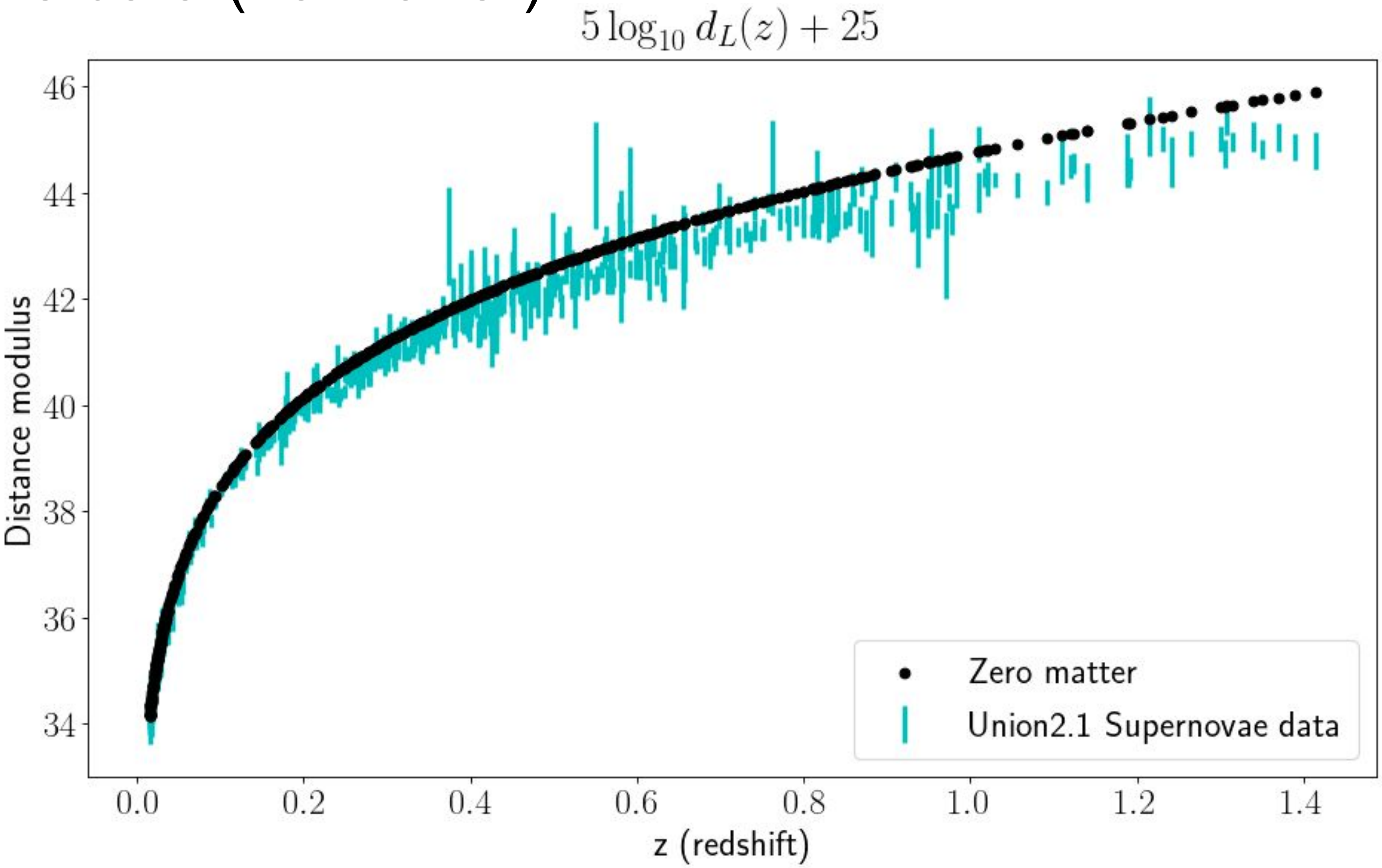
χ^2 , degrees of freedom and standard deviation

$\Delta\chi^2$ above the best fit χ^2 and the confidence intervals for upto 3 degrees of freedom

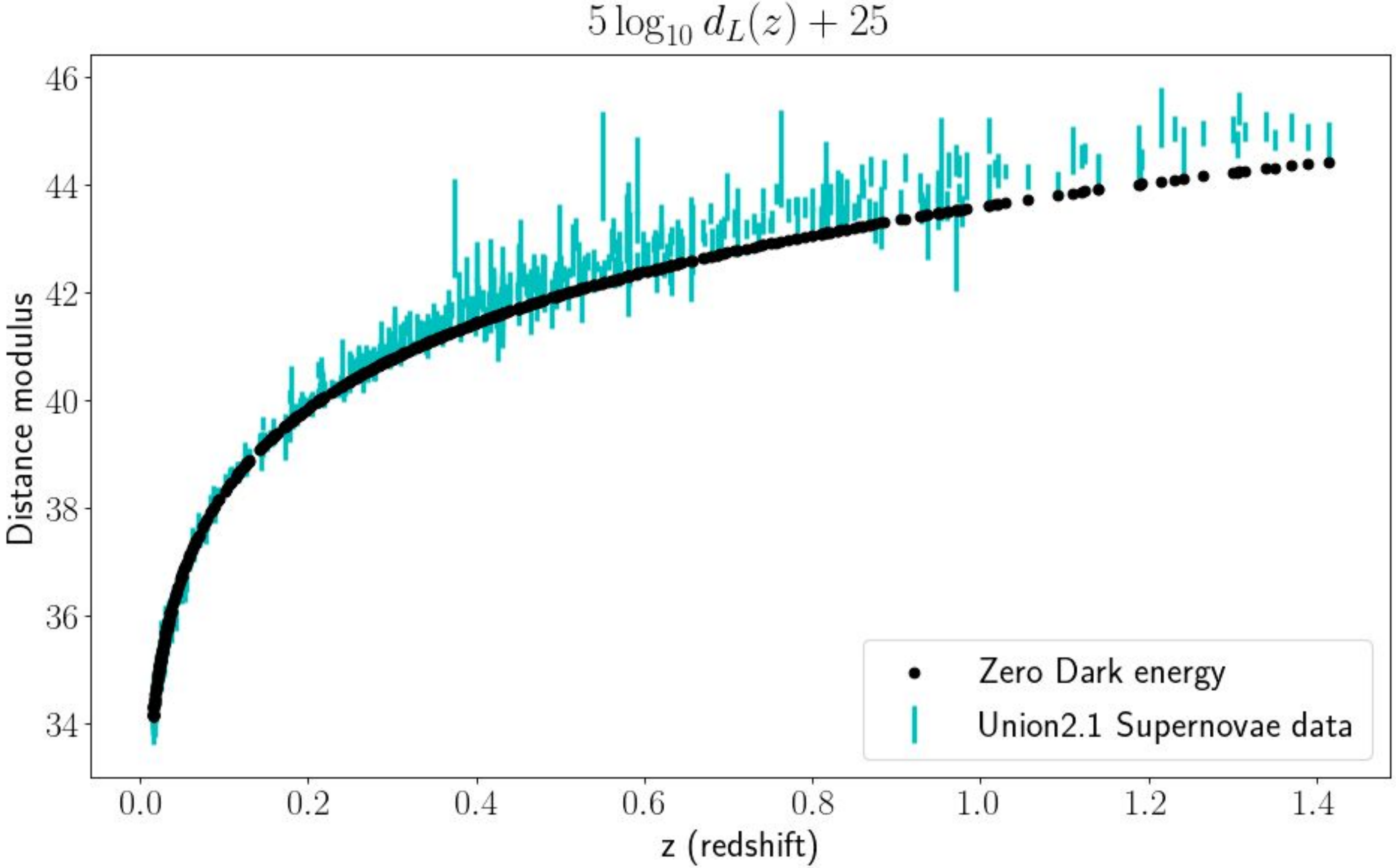
significance	Dof: 1	2	3
68%	1	2.3	3.5
90%	2.71	4.61	6.25
99%	6.63	9.21	11.3

The Institute of Mathematical Sciences

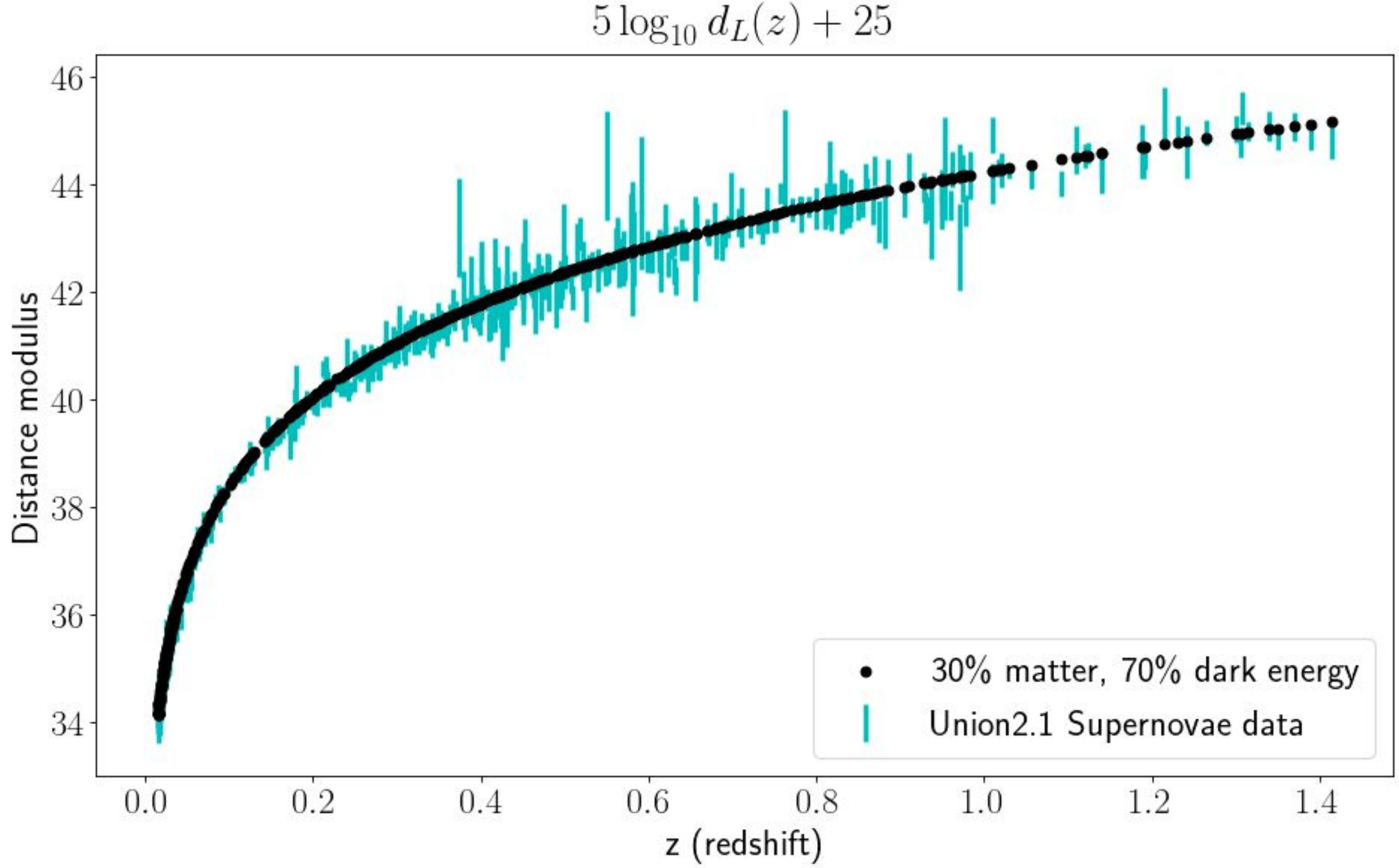
Fit to the data (no matter)



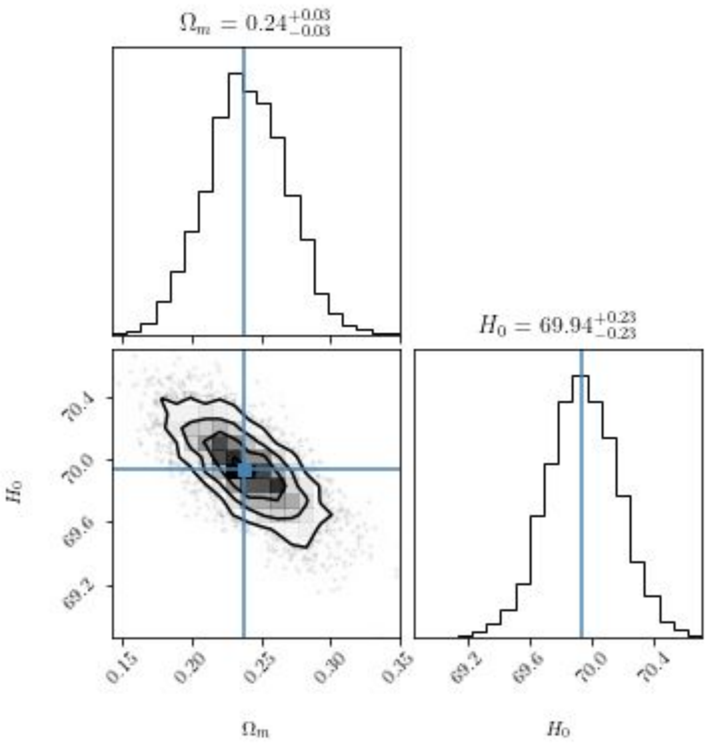
Fit to the data (No dark energy)



Fit to the data (Planck baseline)



Constraints



Constraint on matter density from SNe data

Confidence ellipses are also provided.

On top, the standard deviations are provided along with the mean.

Note the constraint on H_0 does not reflect actual constraint as it is degenerate with the absolute magnitude