November 26, 2022

Mini School on Gravitation and Cosmology

Lecture - II

Dhiraj Kumar Hazra, IMSc, Chennai

The Institute of Mathematical Sciences

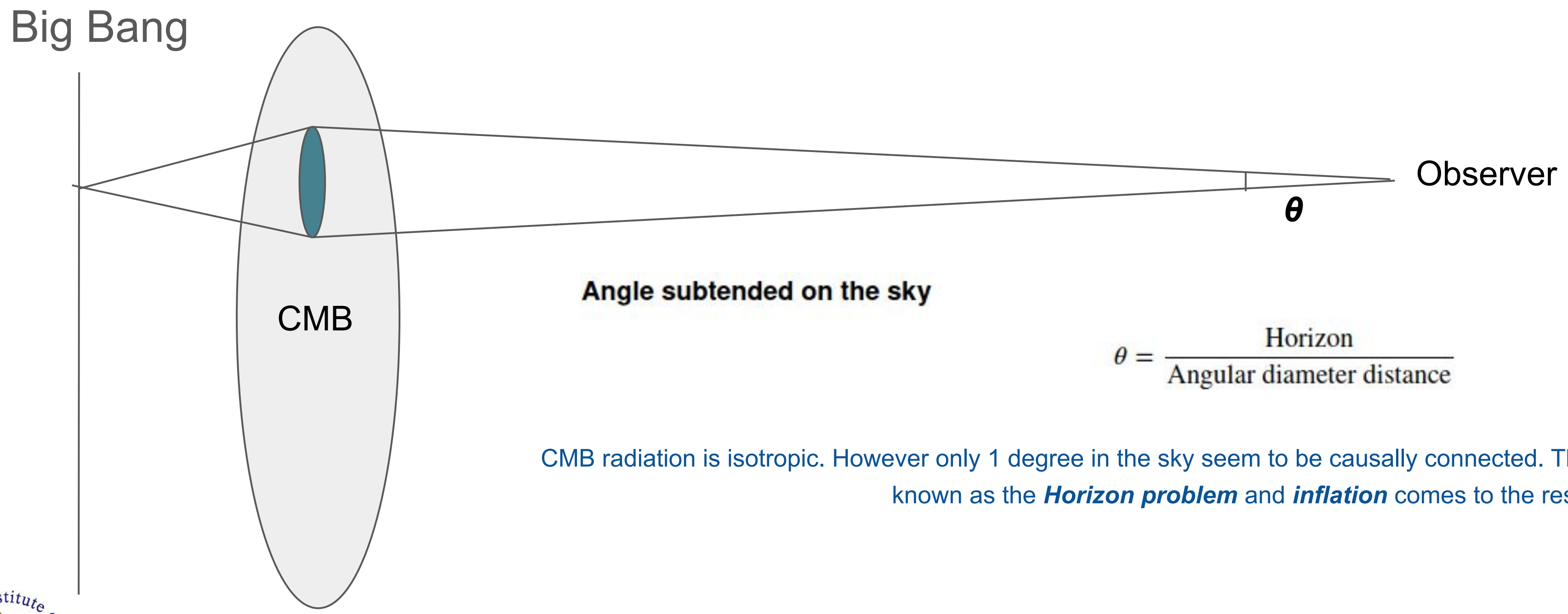
What we will learn

Horizon problem and one of its solution

Scalar field solution

Some comments on the perturbation theory (if possible)

Horizon problem

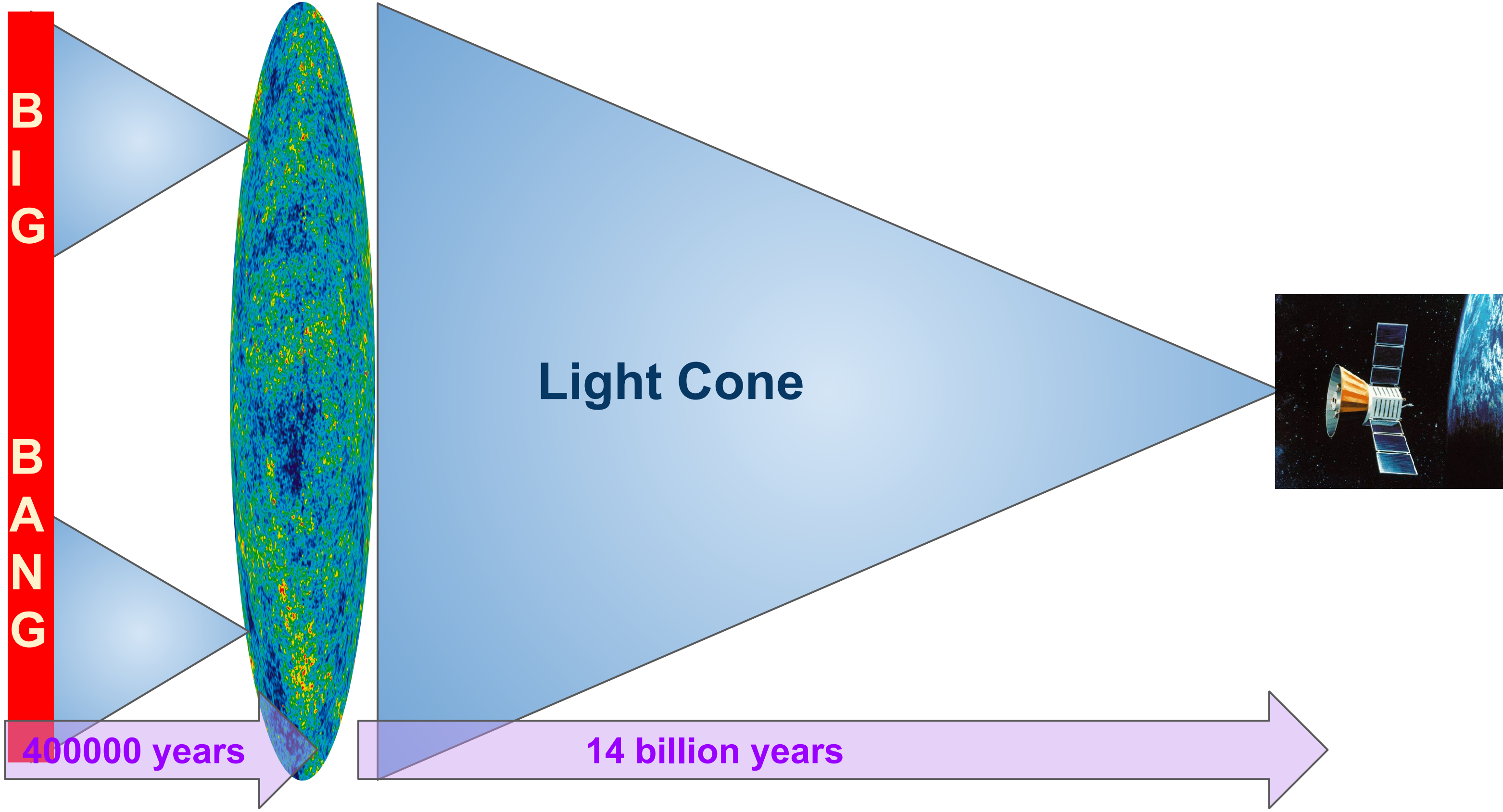


Angle subtended on the sky

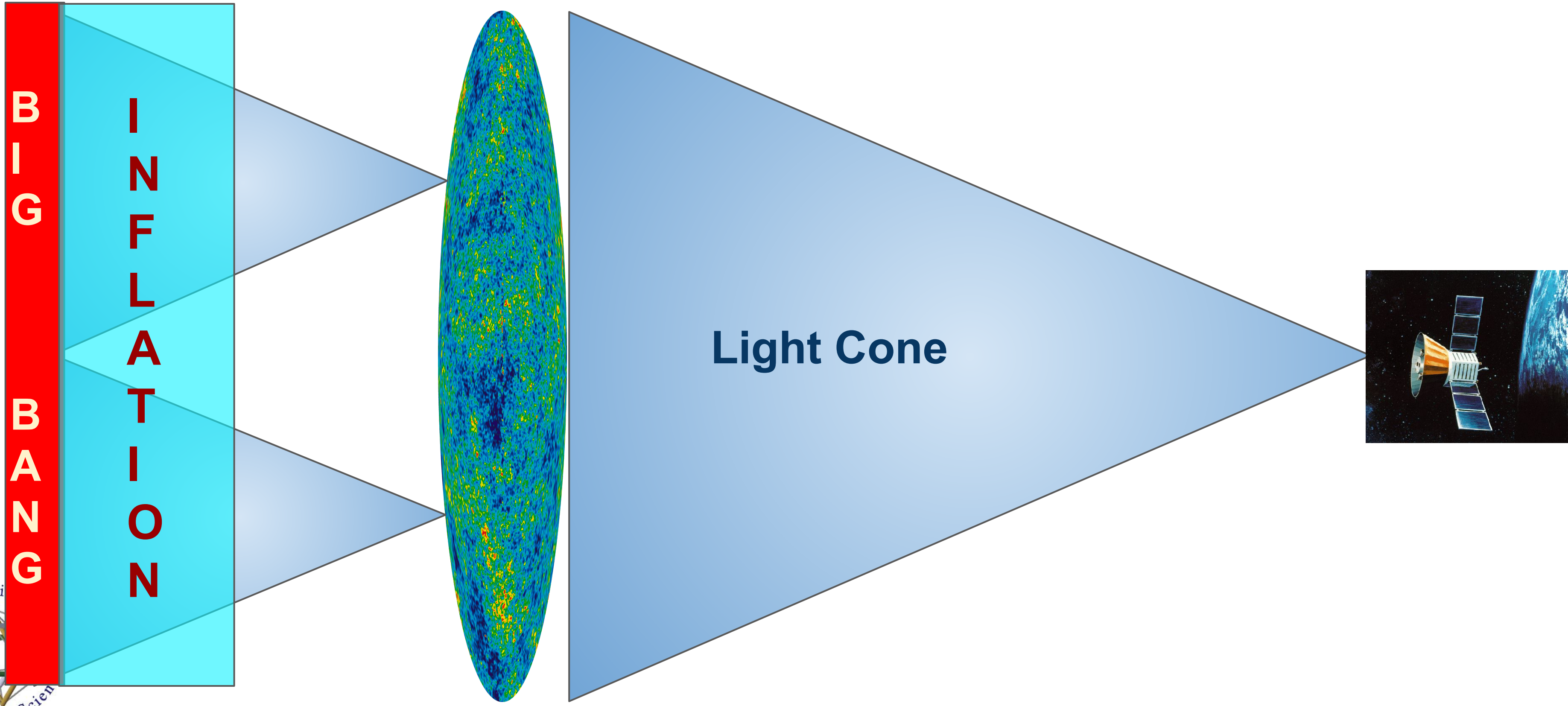
$$\theta = \frac{\text{Horizon}}{\text{Angular diameter distance}}$$

CMB radiation is isotropic. However only 1 degree in the sky seem to be causally connected. This is known as the ***Horizon problem*** and ***inflation*** comes to the rescue.

Cones do not match (Horizon problem)

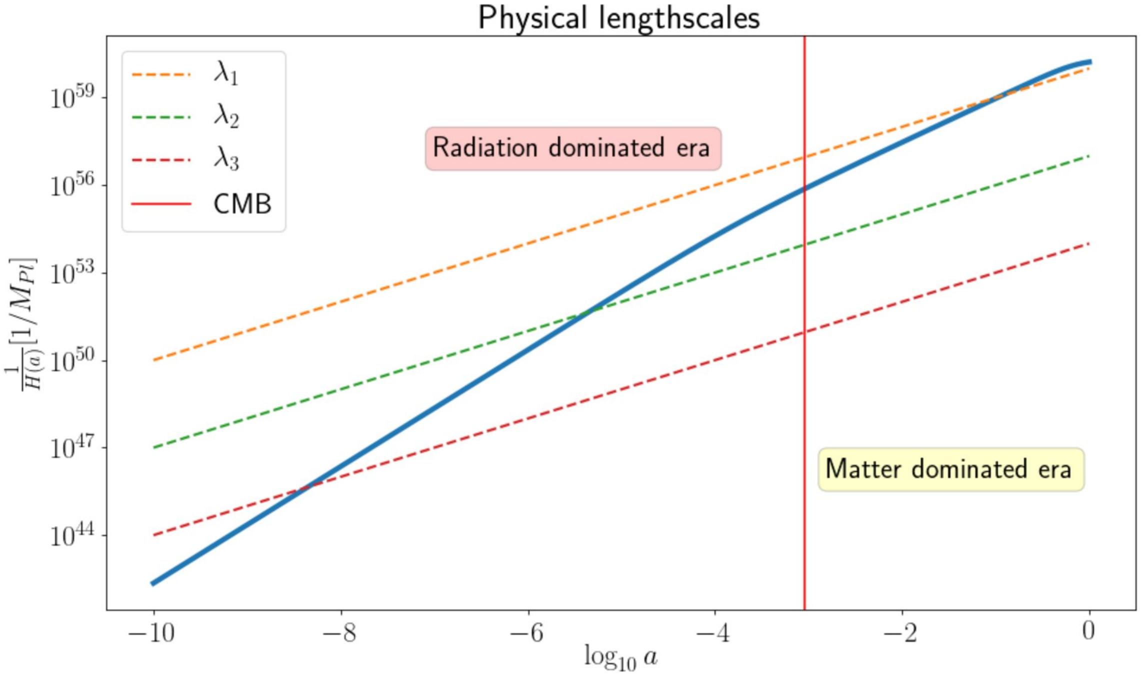


Cones do not match (Horizon problem)

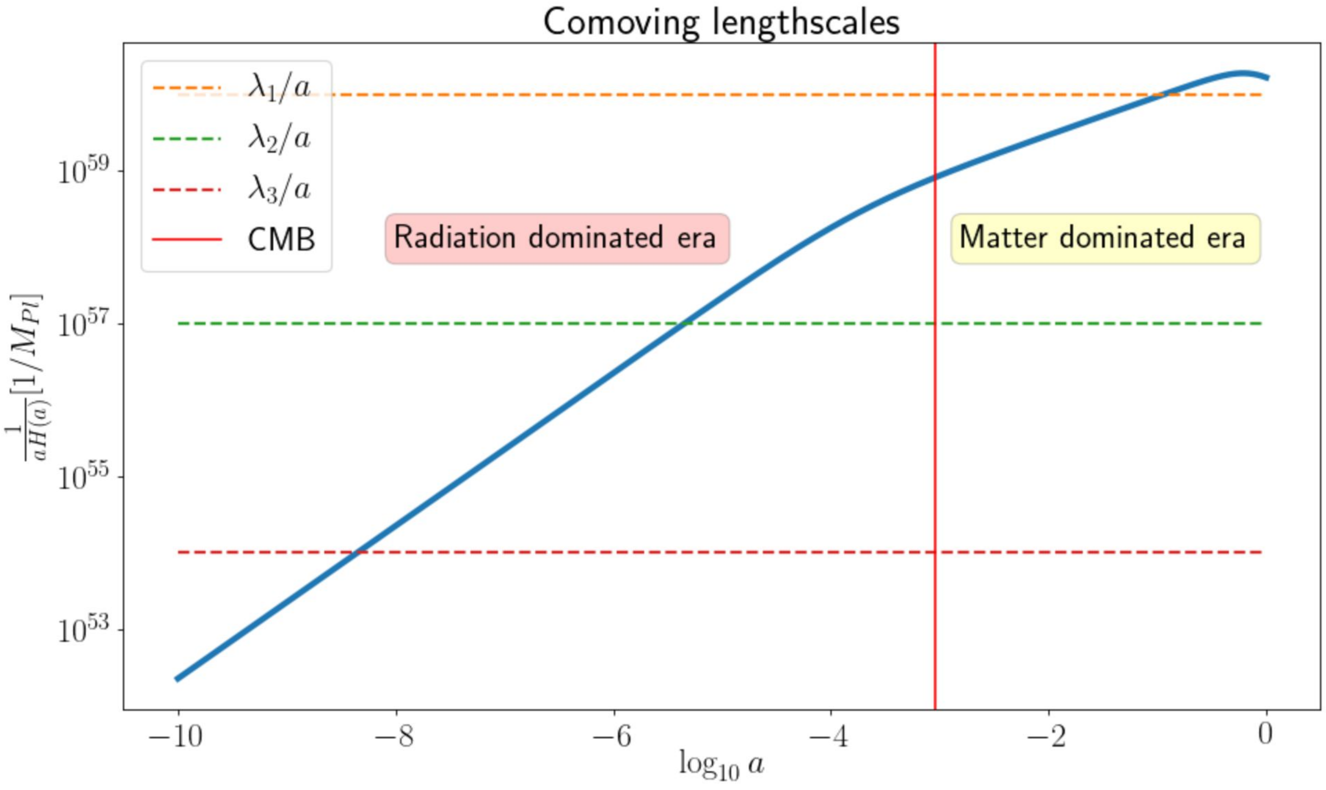


Modes

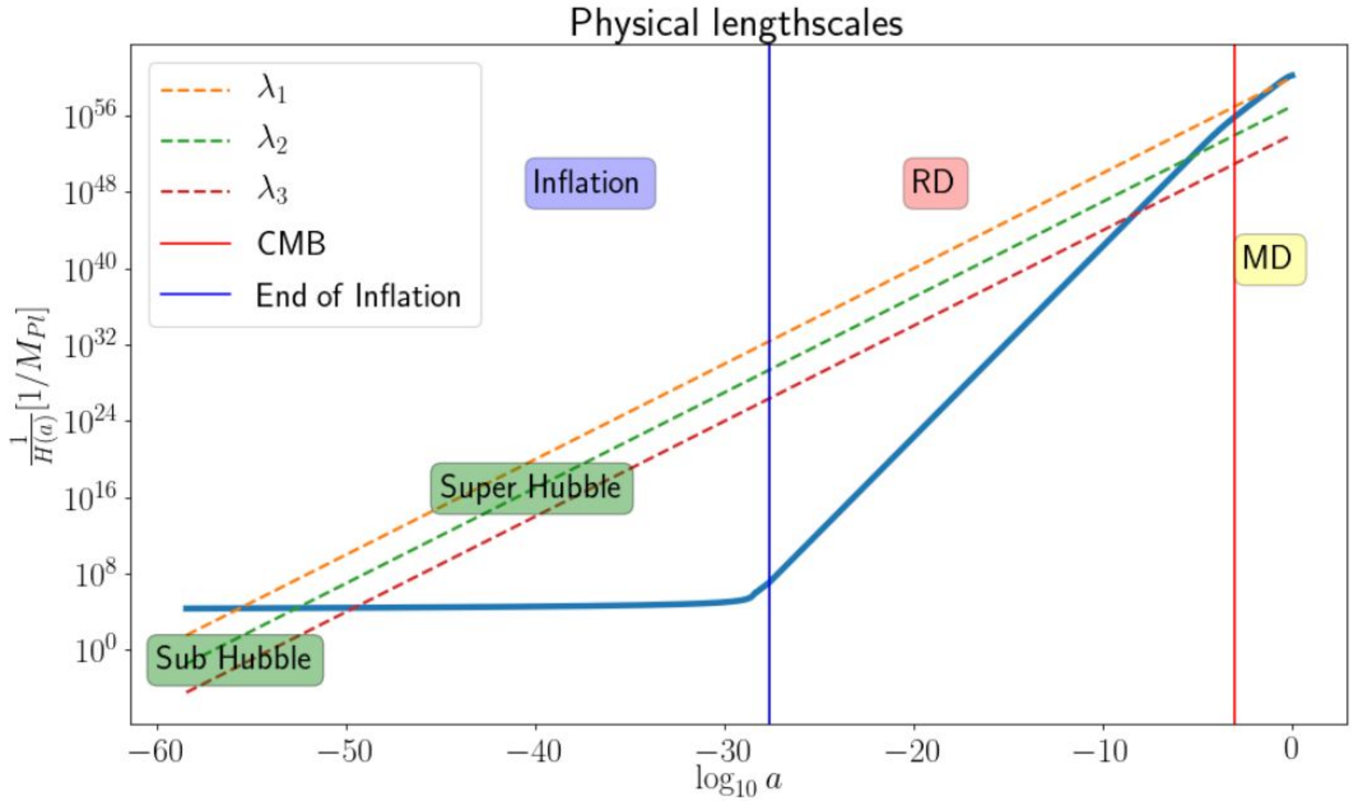
- **Modes leaving the Hubble scale**: A plot of $\log \lambda_p$ versus $\log a$ showing the modes leaving the Hubble radius $d_H = H^{-1}$ at a sufficiently early time



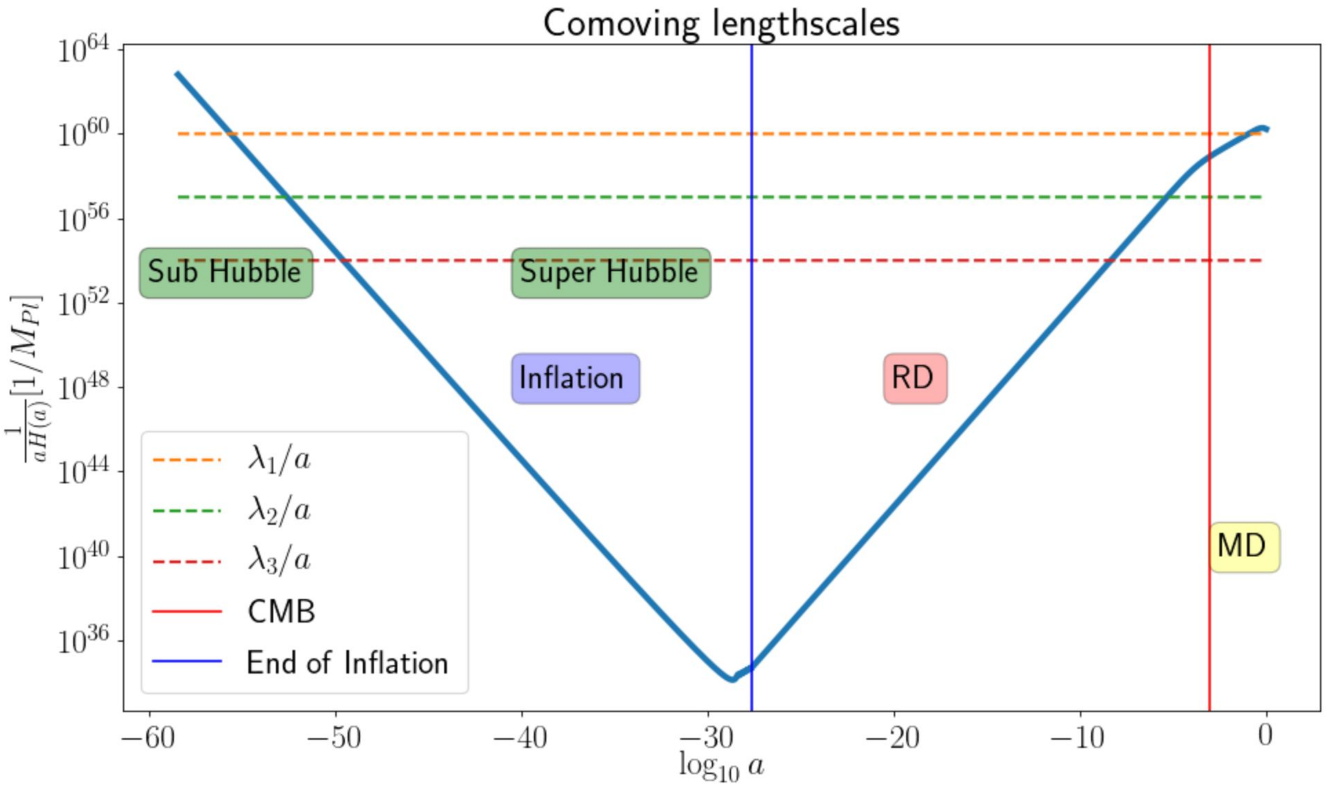
Modes


Comoving lengthscales
λ_1/a
λ_2/a
λ_3/a
CMB
Radiation dominated era
Matter dominated era
10^{59}
10^{57}
10^{55}
10^{53}
$\frac{1}{aH(a)}[1/M_{Pl}]$
−10
−8
−6
−4
−2
0
$\log_{10} a$

Solving horizon problem -- Inflation



Solving horizon problem -- Inflation



Inflation (Scalar fields)

For FRW universe,the density and pressure term for the scalar field(ϕ) can be written as,

$$\rho = (\frac{\dot{\phi}^2}{2} + V)$$

$$p = (\frac{\dot{\phi}^2}{2} - V)$$

Equation of motion of the scalar field:

$$\ddot{\phi} + 3H\dot{\phi} + V_{\phi} = 0$$

where $V_{\phi} = \frac{dV}{d\phi}$

Inflation (necessary conditions)

To satisfy the condition for inflation we need

$$(\rho + 3p) < 0$$

For the scalar field this reduces to

$$\dot{\phi}^2 < V$$

In other words inflation can be achieved if the potential energy dominates the kinetic energy.
If we consider the field to be *slowly rolling* $i.e.$

$$\dot{\phi}^2 << V$$

then it proves to be a sufficient condition for inflation
To resolve horizon problem, we need at least ~ 60 e-folds.So the field should slow roll over a sufficiently long period of time

$$\ddot{\phi} << 3H\dot{\phi}$$

How to solve these equations numerically

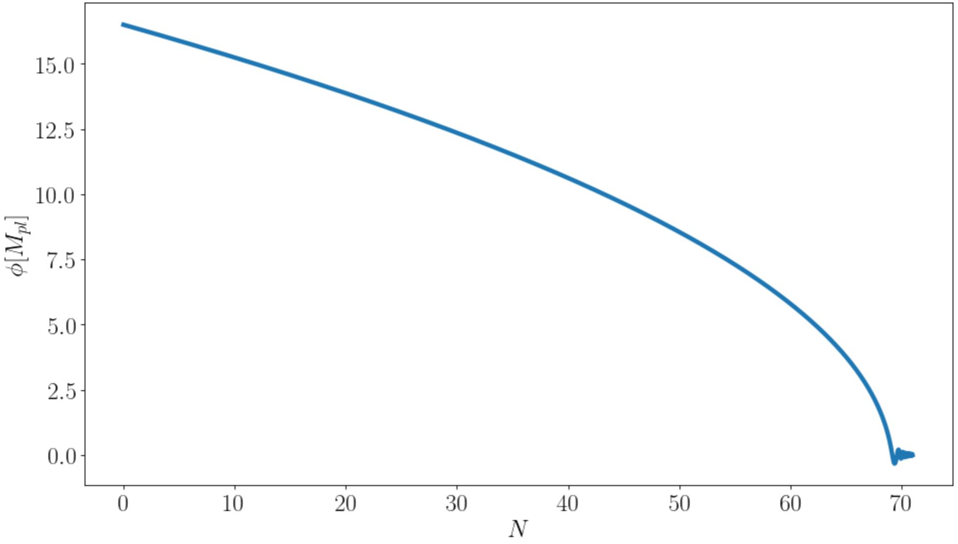
Approximation fails in many cases:

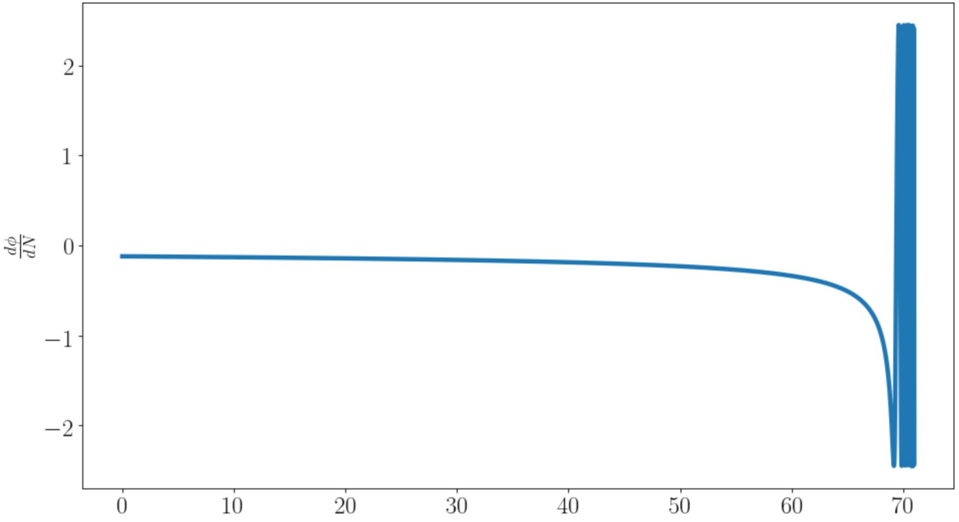
$$\frac{dy_i(x)}{dx} = f_i(x, y_1, \ldots, y_N), \qquad i = 1, \ldots, N$$

Euler method:

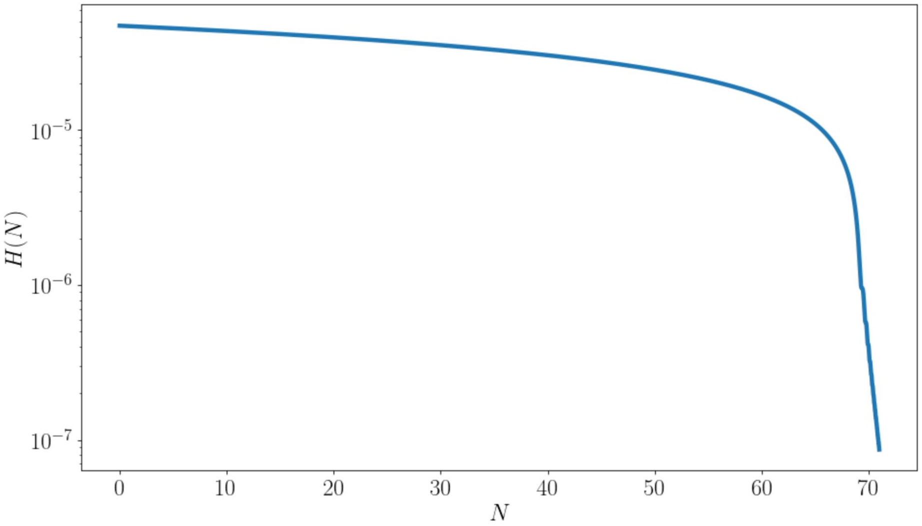
$$y_{n+1} = y_n + hf(x_n, y_n)$$

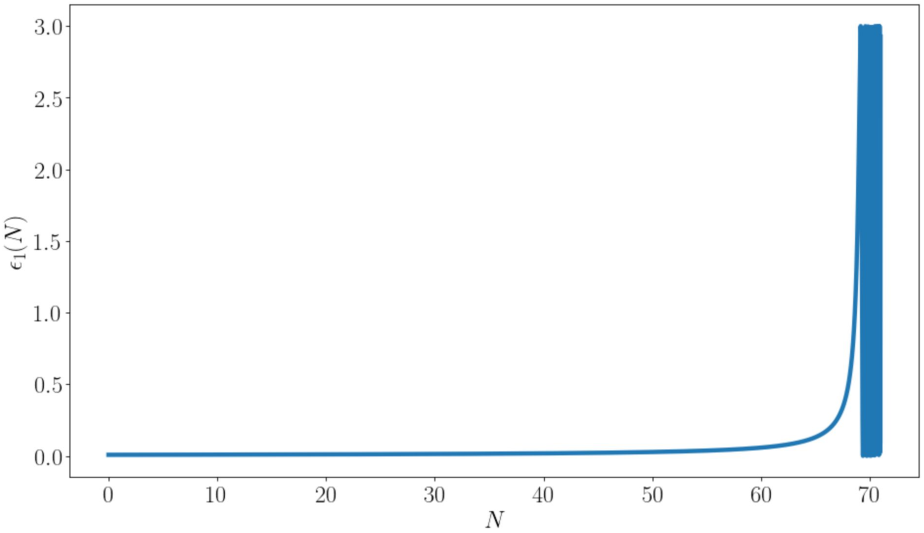
Inflation: scalar field evolution





Inflation: slow roll





The Institute of Mathematical Sciences

Metric perturbation (Linear order)

$$G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu}$$

where,

$$G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2} R \delta^{\mu}_{\nu}$$

Introduce perturbation in FRW metric in (n+1) dimensions:

$$ds^2 = (1+2A)dt^2 - 2a(\partial_i B + S_i)dtdx^i - a^2(t)[(1-2\Psi)\delta^i_j + 2\partial_i\partial_j E + (\partial_i F_j + \partial_j F_i) + h_{ij}]dx^i dx^j$$

Degrees of freedom

Scalars=(A, B, Ψ, E)=4

Vectors=$\overrightarrow{F}$, $\overrightarrow{S}$=2(n-1)

Tensors=h_{ij}=$\frac{n(n+1)}{2}$

The Institute of Mathematical Sciences

Metric perturbation (Linear order)

Will not be discussing degrees of freedom, coordinate transformation and choices of Gauges. See Inflation reviews.

Only one scalar function in metric needed: $A = \Psi = \Phi$

$$3H(H\Phi + \dot{\Phi}) - (1/a^2)\nabla^2\Phi = -(4\pi G)\delta\rho$$

$$H\Phi + \dot{\Phi} = 4\pi G\delta q$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (2\dot{\Phi} + 3H^2)\Phi = 4\pi G\delta p$$

$\delta p = C_A^2\delta\rho + \delta p_{NA}$ (Non adiabatic pressure pert.)

Next Step: Go to conformal time coordinate. Obtain a single scalar perturbation equation.

Bardeen equation

$$\Phi'' + 3\mathcal{H}(1 + C_A^2)\Phi' - C_A^2\nabla^2\Phi + [2\mathcal{H}' + (1 + 3C_A^2\mathcal{H}^2)]\Phi = (4\pi G a^2)\delta p_{NA}$$

where,

$$\mathcal{H} = \frac{a'}{a} \text{ The Conformal Hubble Parameter}$$

$$C_A^2 = (p'/\rho')$$

C_A^2=Adiabatic speed Of perturbation

Curvature perturbation (Conserved at SHS)

Define ,

$$\mathcal{R} = -\frac{1}{\mathcal{H}^2 - \mathcal{H}'}[\mathcal{H}\Phi' + (2\mathcal{H}^2 - \mathcal{H}')\Phi]$$

According to the Bardeen equation $\mathcal{R}$ satisfies,

$$\mathcal{R}' = -\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}[(4\pi G a^2)\delta p_{NA} + C_A^2 \nabla^2 \Phi]$$

Neglect non-adiabatic case $\Rightarrow \delta p_{NA} = 0$, go to fourier space,

$$\mathcal{R}' = \frac{\mathcal{H} C_A^2}{\mathcal{H}^2 - \mathcal{H}'}(k^2 \Phi)$$

At super-Hubble limit, $k \to 0$ and we have $\mathcal{R}' \simeq 0$.
For adiabatic case, $\mathcal{R}_k$ is conserved at super-Hubble scales.

The Institute of Mathematical Sciences

Curvature perturbation (Conserved at SHS)

Using the formula for the δp_{NA} , $\mathcal{R}'$ becomes

$$\mathcal{R}' = \left(\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\right)(\nabla^2\,\Phi)$$

Combine $\mathcal{R}'$ equation with the Bardeen Equation,
$\Longrightarrow$ The equation for The curvature perturbation:

$$\boxed{\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' - \nabla^2\mathcal{R} = 0}$$

where z is defined as,

$$z = \frac{a\phi'}{\mathcal{H}}$$

In the Fourier k-space

$$\boxed{\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0}$$

Mukhanov Sasaki Equations

Define v as $v = \mathcal{R}z$
after substitution we get

$$\boxed{v_k'' + \left[k^2 - \frac{z''}{z}\right] v_k = 0}$$

This is Mukhanov-Sasaki equation

$$\lim_{(k/\mathcal{H})\to\infty} v_k(\eta) \to \left(\frac{1}{\sqrt{2k}}\right) \mathrm{e}^{-ik\eta}$$

This is the initial condition of the equation at sub-Hubble scales known as Bunch-Davies initial condition

Power spectrum and its tilt

The Power Spectrum of the scalar perturbations($\mathcal{P}_S(k)$)

$$\mathcal{P}_S(k) = \left(\frac{k^3}{2\pi^2}\right) |\mathcal{R}_k|^2 = \left(\frac{k^3}{2\pi^2}\right) \left(\frac{|v_k|}{z}\right)^2$$

The $\mathcal{R}_k$ is evaluated at super-Hubble scales $i.e.$ at $(k/\mathcal{H}) << 1$

Scalar spectral index is defined as

$$n_s = 1 + \left(\frac{d\ln \mathcal{P}_s}{d\ln k}\right)$$

Slow roll

I am following notations consistent to Sriramkumar's review 2009.

$$z = \sqrt{2}\, M_{\rm P}\ \left(a\,\sqrt{\epsilon_{\rm H}}\right)$$

The slow roll parameters:

$$\epsilon_{\rm H} = 1 - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right) \quad \text{and} \quad \delta_{\rm H} = \epsilon_{\rm H} - \left(\frac{\epsilon_{\rm H}'}{2\,\mathcal{H}\,\epsilon_{\rm H}}\right)$$

$$\left(\frac{z''}{z}\right) = \mathcal{H}^2 \left[2 - \epsilon_{\rm H} + (\epsilon_{\rm H} - \delta_{\rm H})\ (3 - \delta_{\rm H}) + \left(\frac{\epsilon_{\rm H}' - \delta_{\rm H}'}{\mathcal{H}}\right)\right] \qquad \left(\frac{a''}{a}\right) = \mathcal{H}^2\ (2 - \epsilon_{\rm H})$$

November 26, 2022

Slow roll

If we parametrize

$$\mathcal{P}_{\rm S}(k) \simeq \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2_{k=(a\,H)},$$

$$\mathcal{P}_{\rm T}(k) \simeq \left(\frac{8}{M_{\rm P}^2}\right)\left(\frac{H}{2\pi}\right)^2_{k=(a\,H)}$$

The Institute of Mathematical Sciences

Slow roll

We obtain the spectral index and tensor to scalar ratio as:

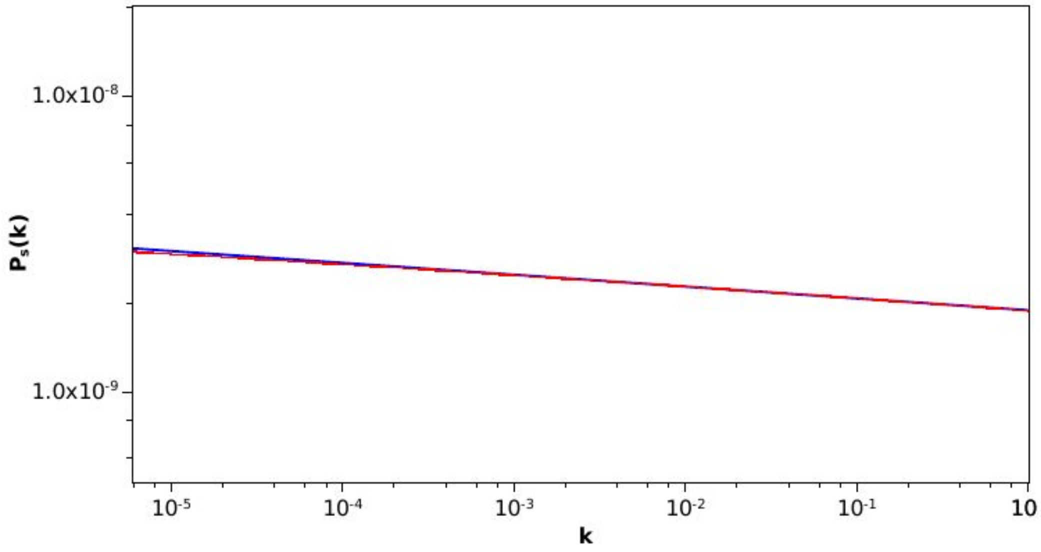
$$n_{\mathrm{S}} \simeq (1 - 4\,\epsilon_{\mathrm{H}} + 2\,\delta_{\mathrm{H}}) \quad \text{and} \quad n_{\mathrm{T}} \simeq -\,(2\,\epsilon_{\mathrm{H}})$$

$$r \simeq (16\,\epsilon_{\mathrm{H}}) = -\,(8\,n_{\mathrm{T}})$$

The relation between the tensor spectral tilt and the tensor to scalar ratio is the consistency relation

Power spectrum and its tilt

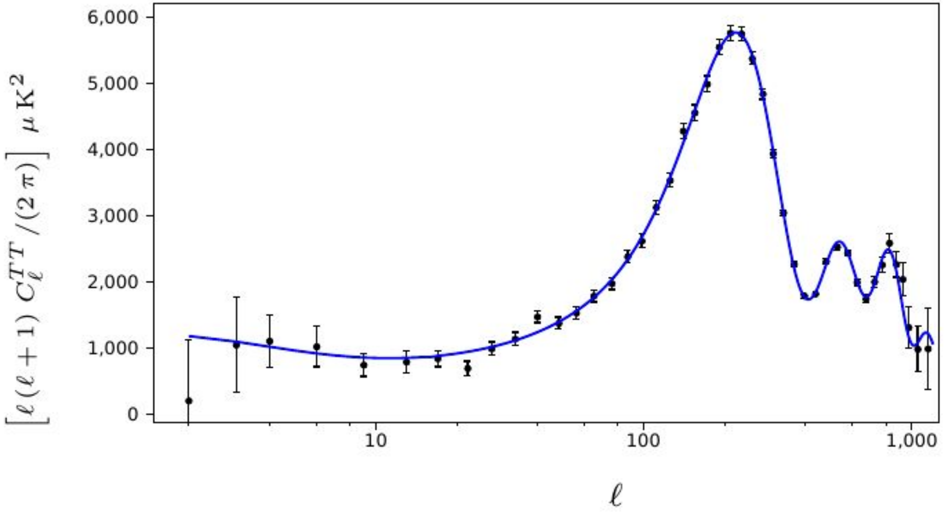
- The power law scalar power spectrum, and the spectrum from the quadratic potential $(m^2\phi^2/2)$ are shown below. They are almost indistinguishable, and they fit the data to the same extent.



The Institute of Mathematical Sciences

Power spectrum and its tilt

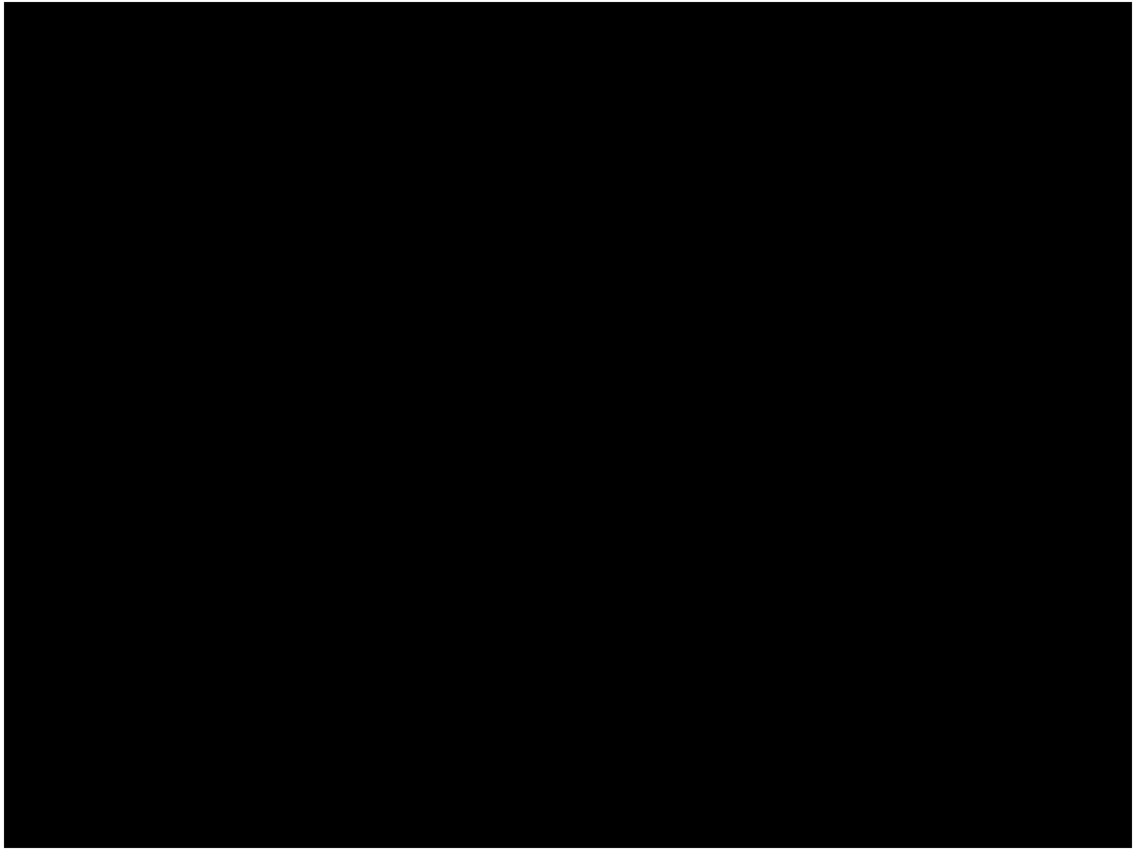
- Standard slow roll inflation produces almost the same angular power spectrum as a power law primordial spectrum
- We have plotted below the CMB angular power spectrum for the best fit values of the canonical scalar field described by the quadratic potential.



The Institute of Mathematical Sciences

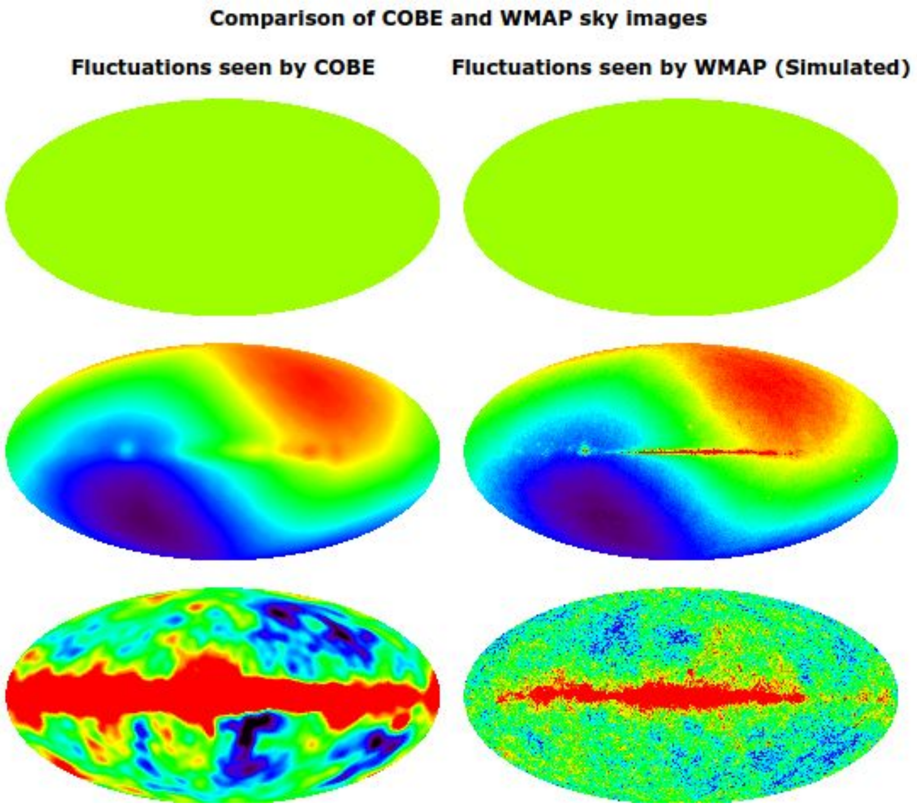
November 26, 2022

Cosmic Microwave Background (Observation)

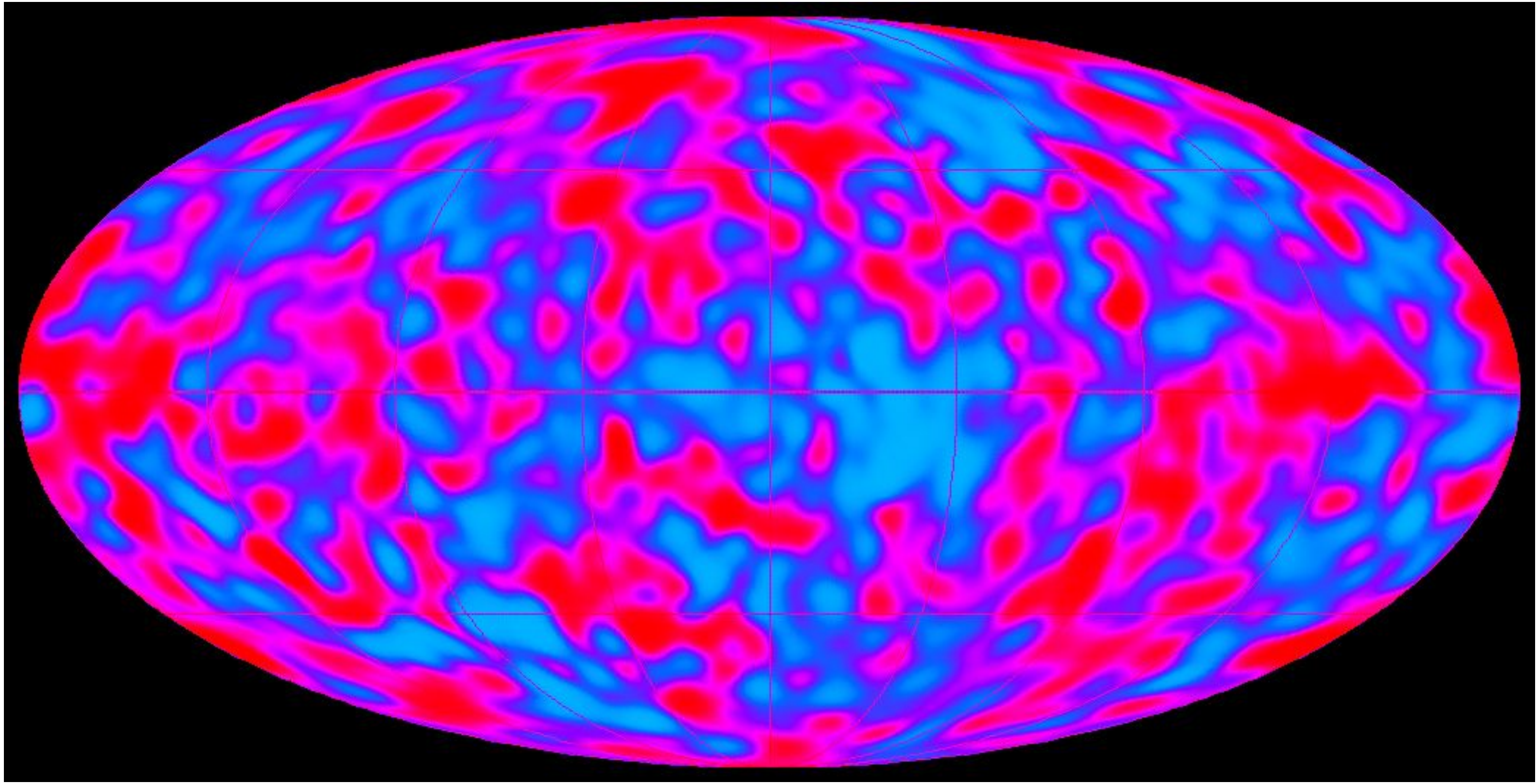


The Institute of Mathematical Sciences

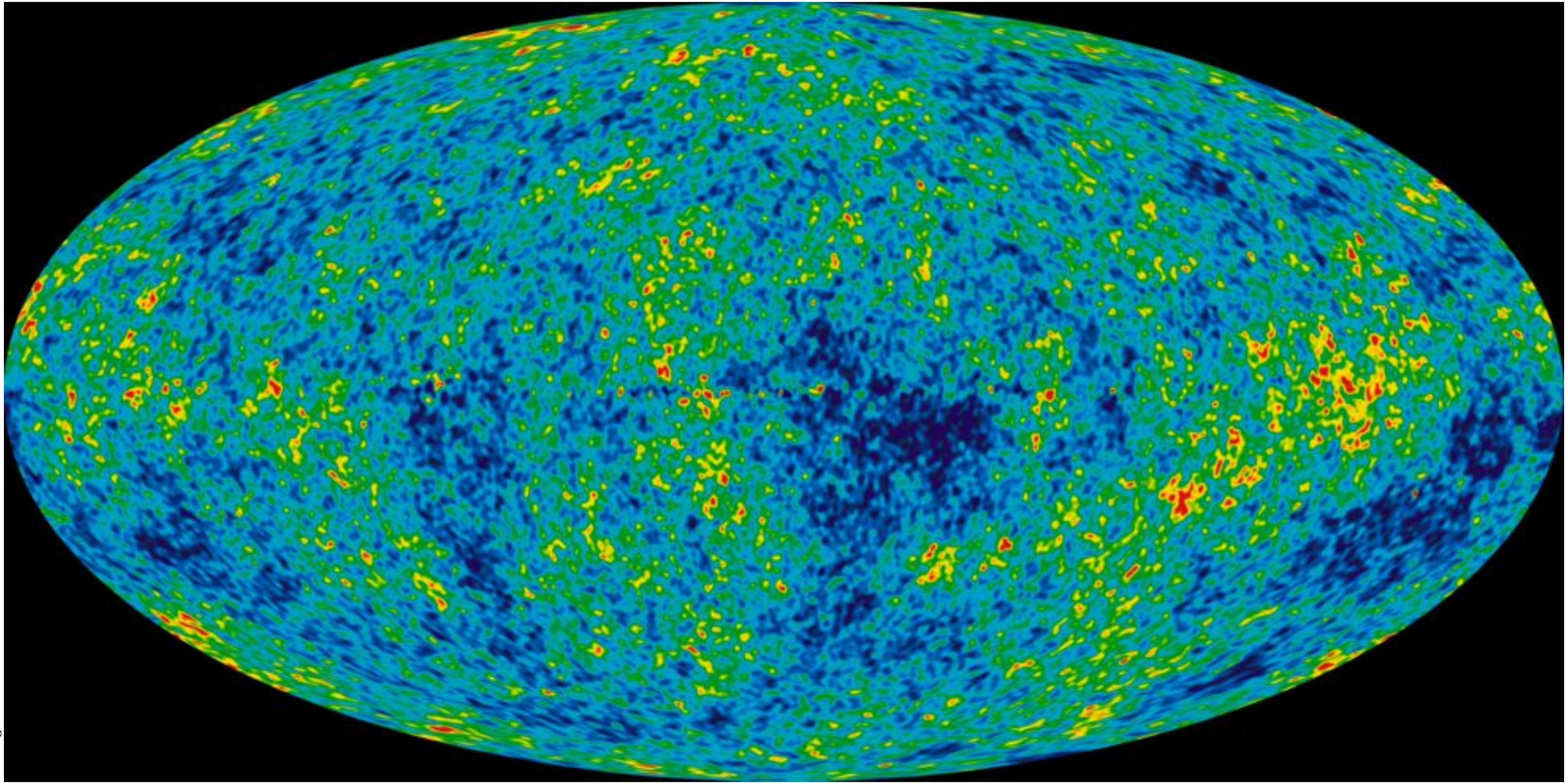
Cosmic Microwave Background (Evolution: COBE)



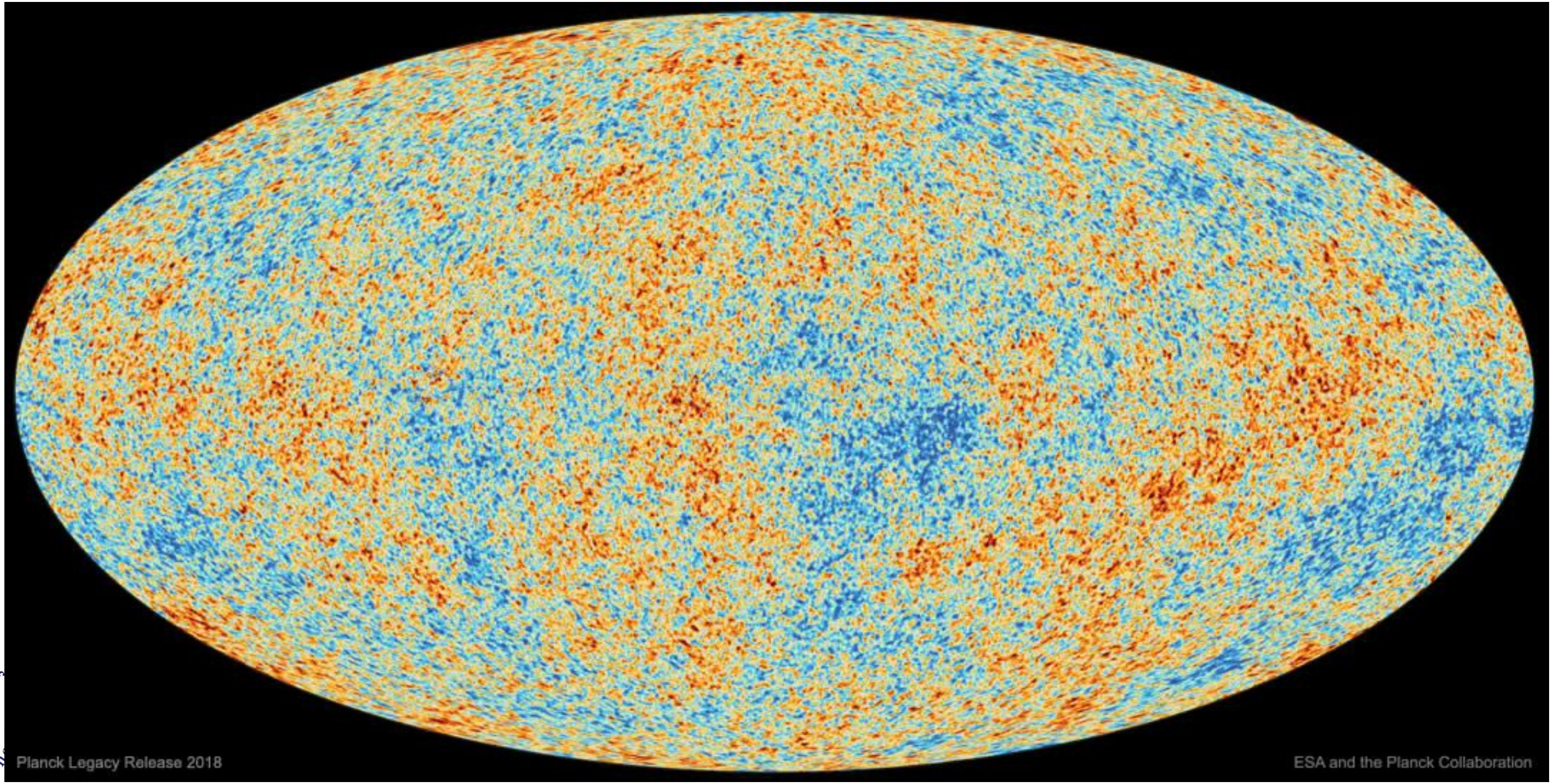
Cosmic Microwave Background (Evolution: COBE)



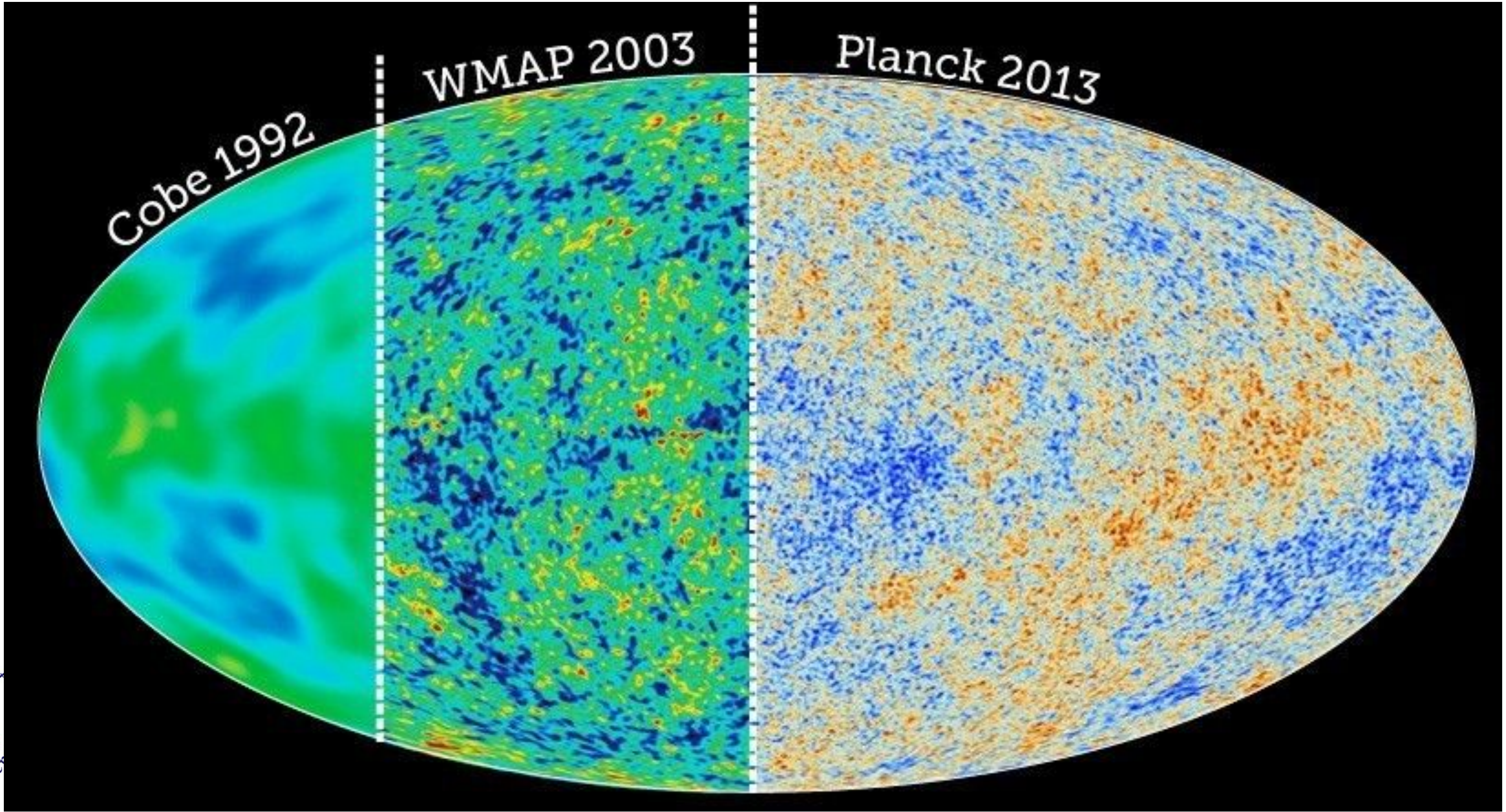
Cosmic Microwave Background (Evolution: WMAP)



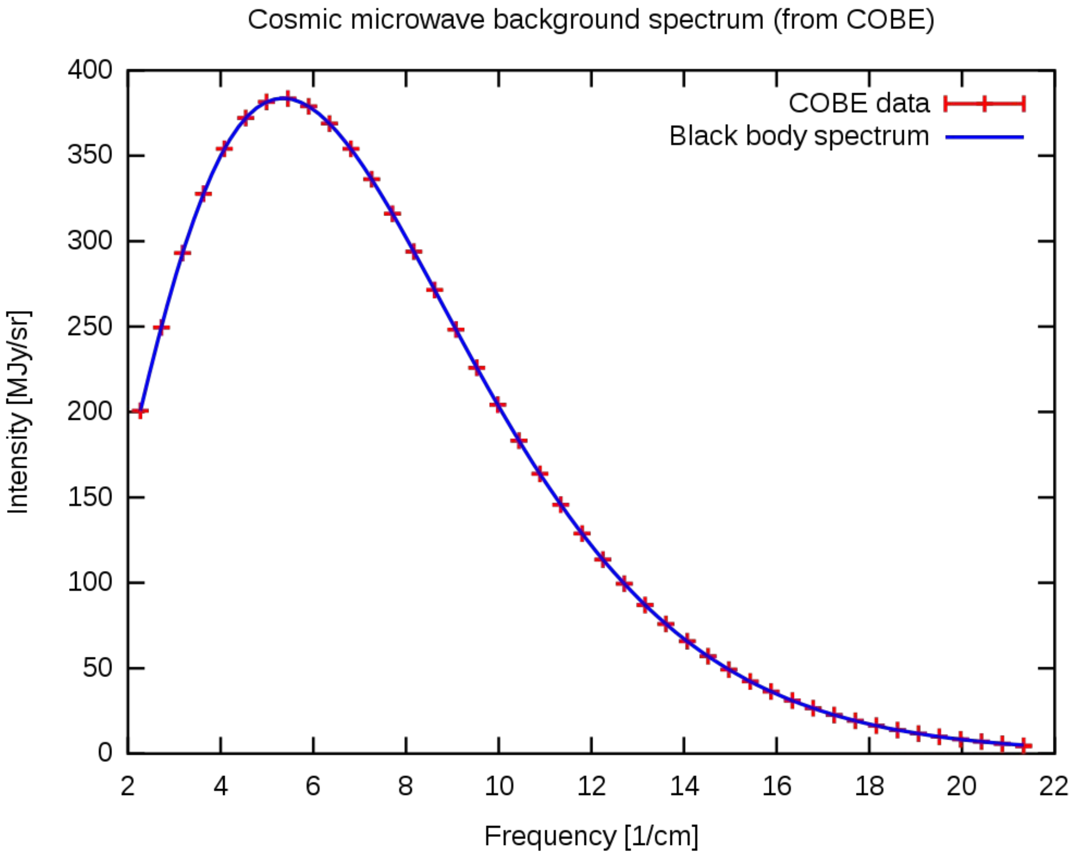
Cosmic Microwave Background (Evolution: Planck)



Cosmic Microwave Background (MAP)

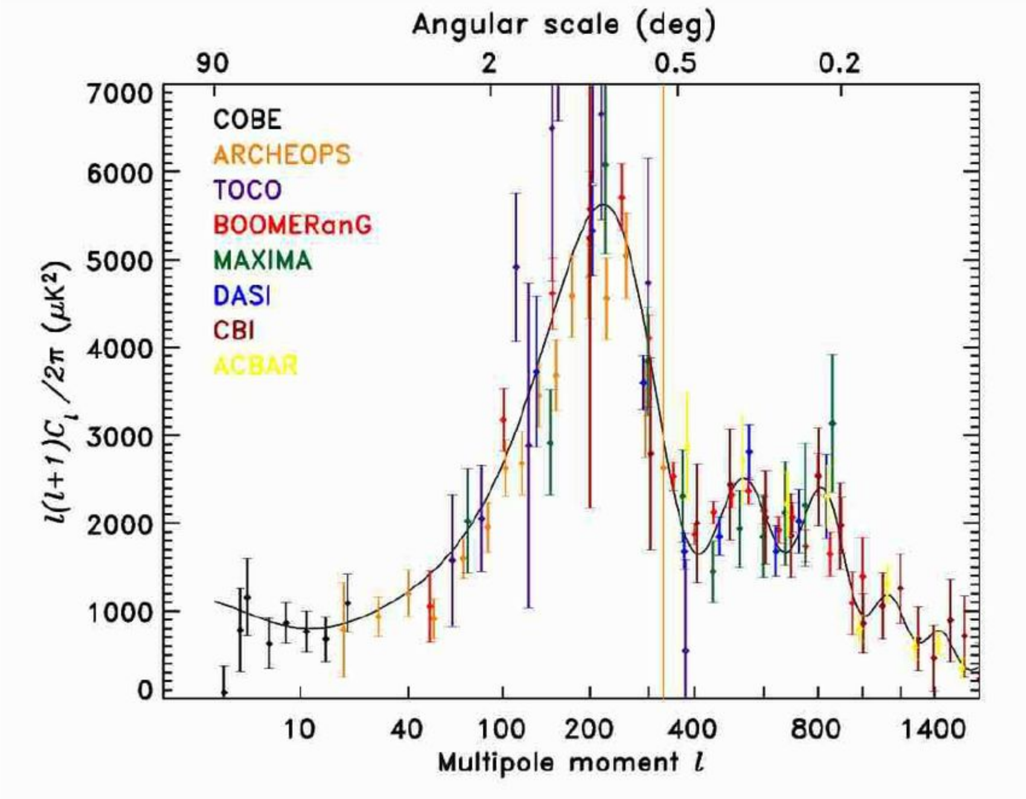


Cosmic Microwave Background (BlackBody)

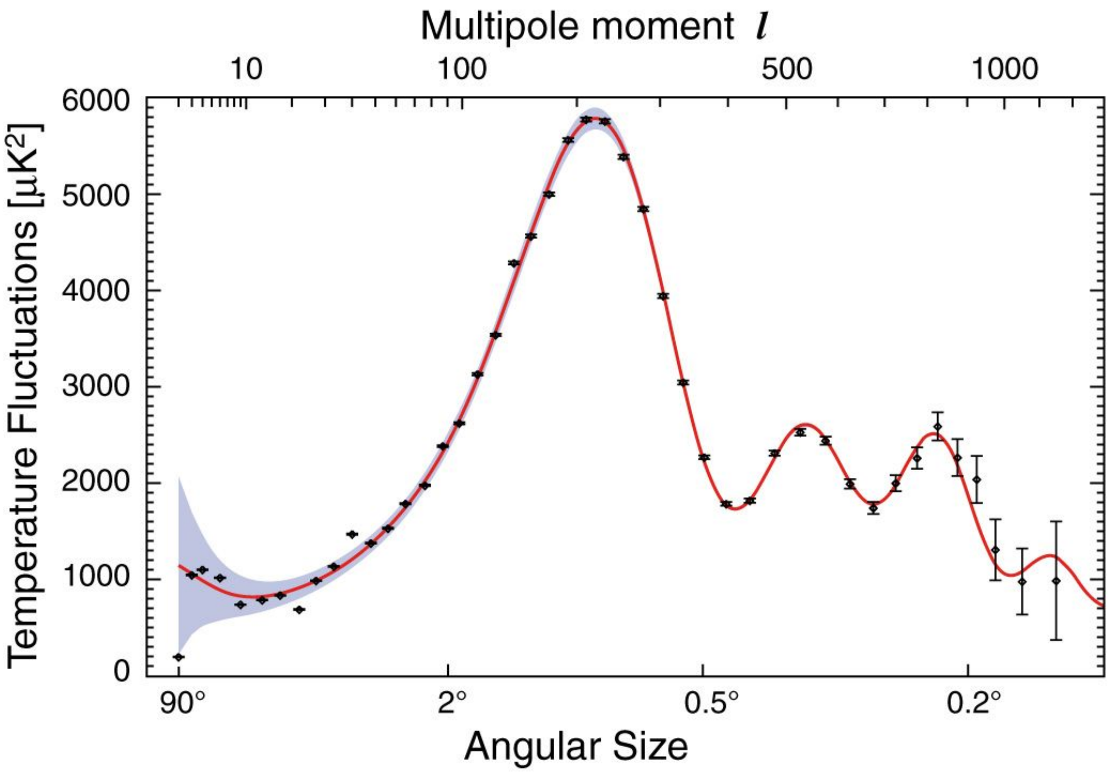


The Institute of Mathematical Sciences

Cosmic Microwave Background (Power spectrum)

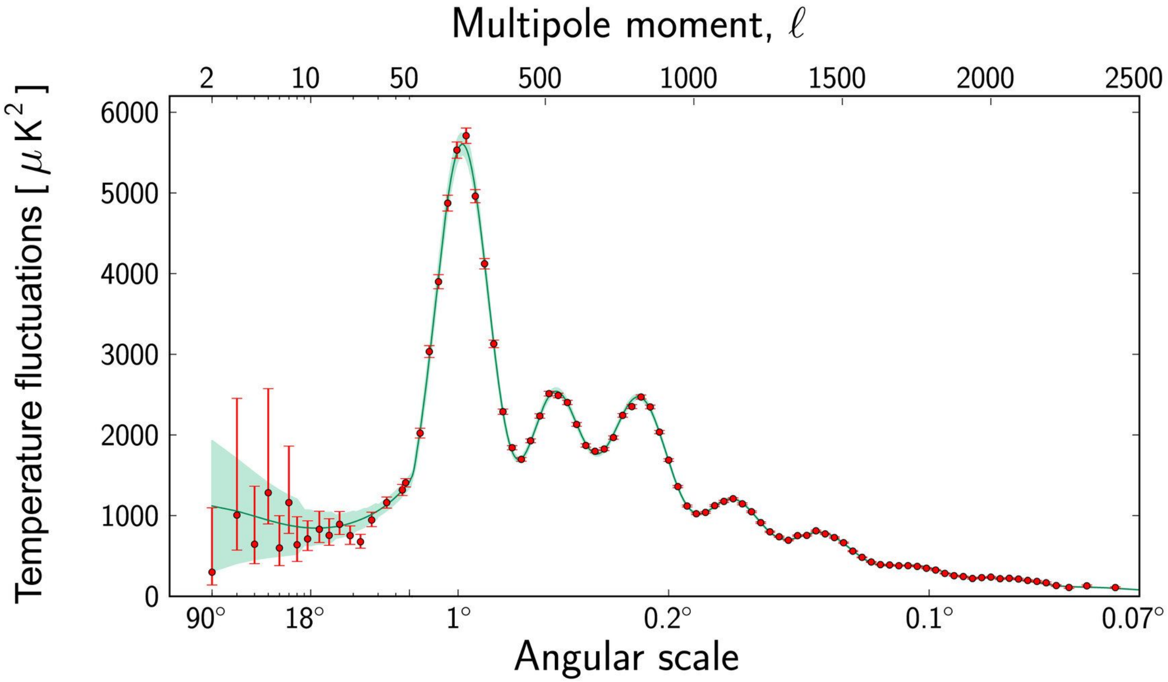


Cosmic Microwave Background (PS WMAP)

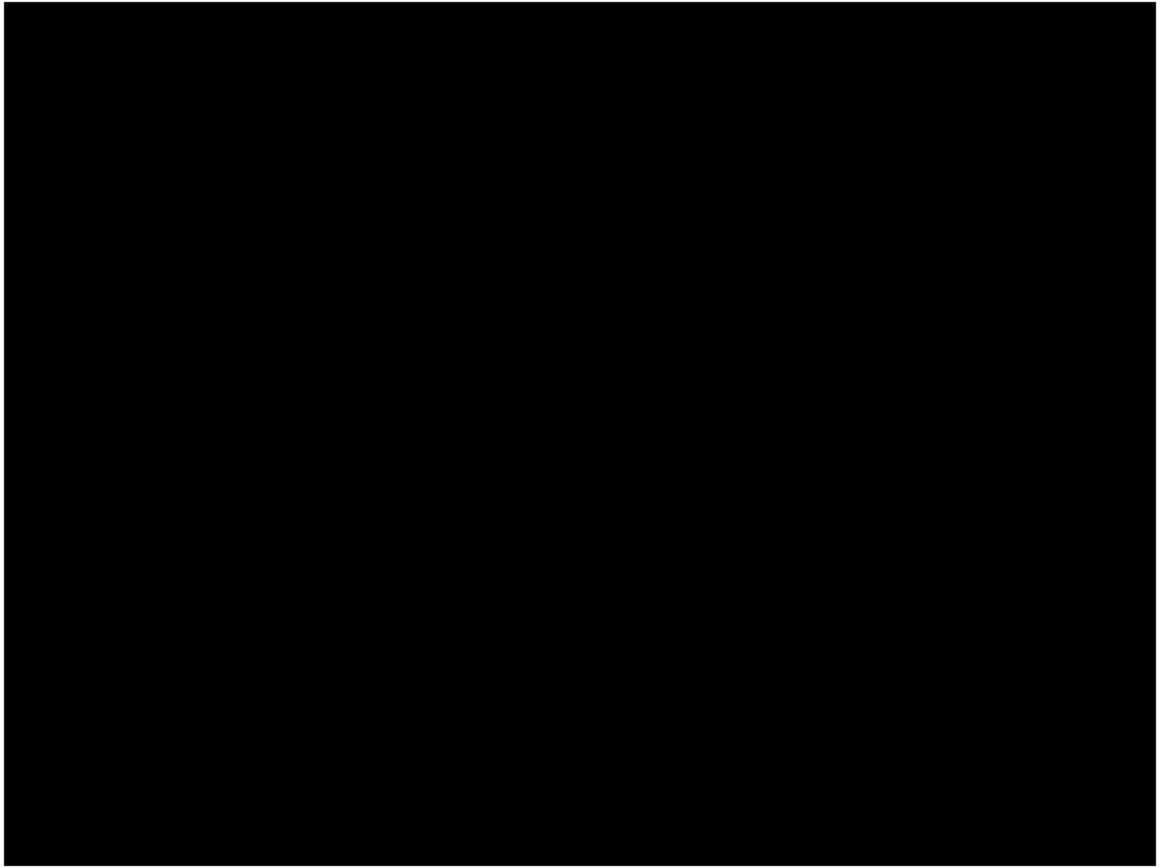


The Institute of Mathematical Sciences

Cosmic Microwave Background (PS Planck)

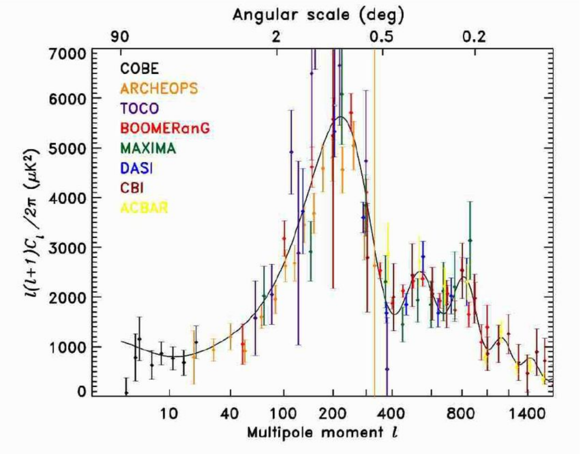


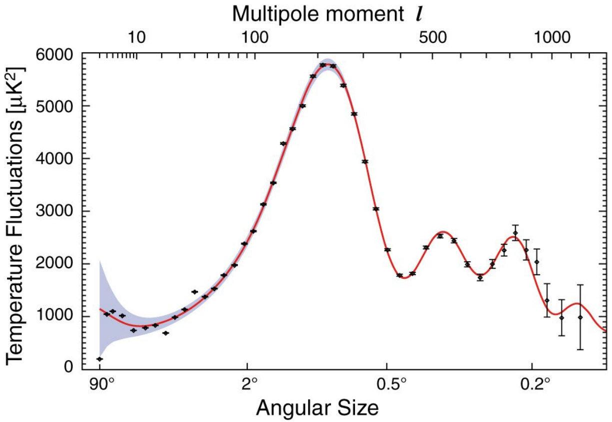
Cosmic Microwave Background (Physics of scales)

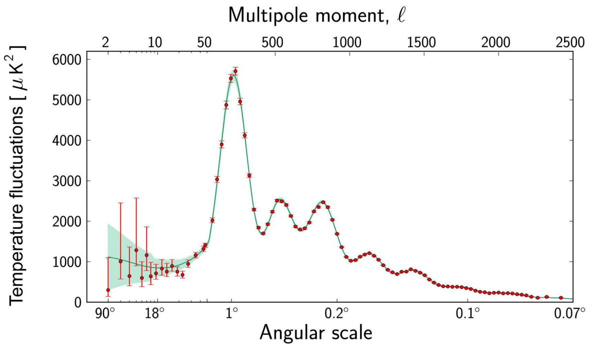


The Institute of Mathematical Sciences

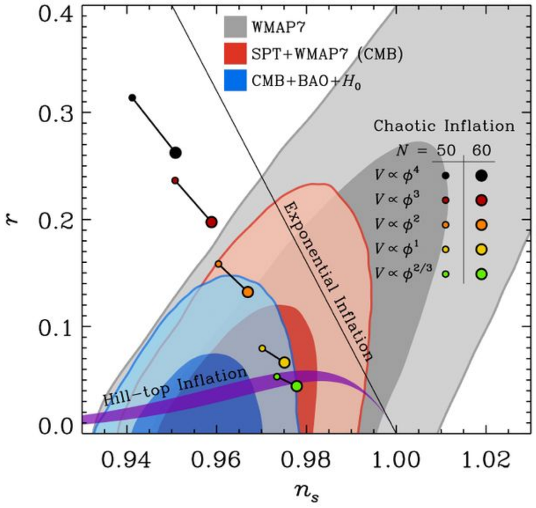
Cosmic Microwave Background (Power spectrum)

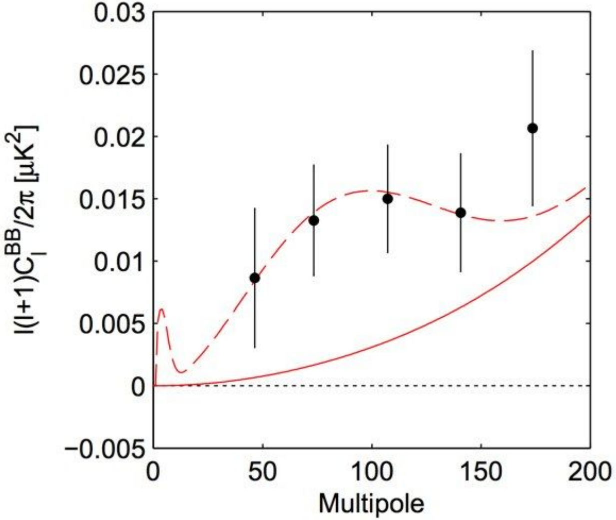


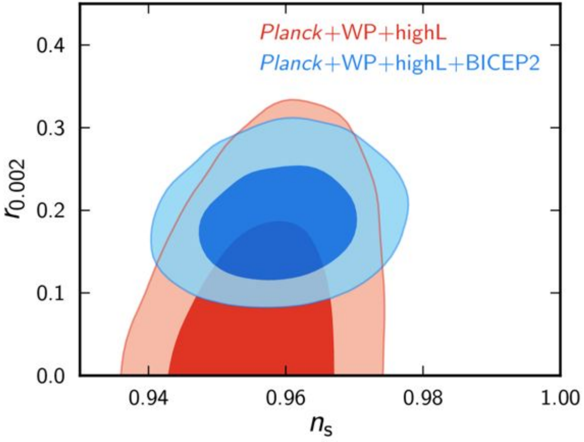




Cosmic Microwave Background (PS WMAP)





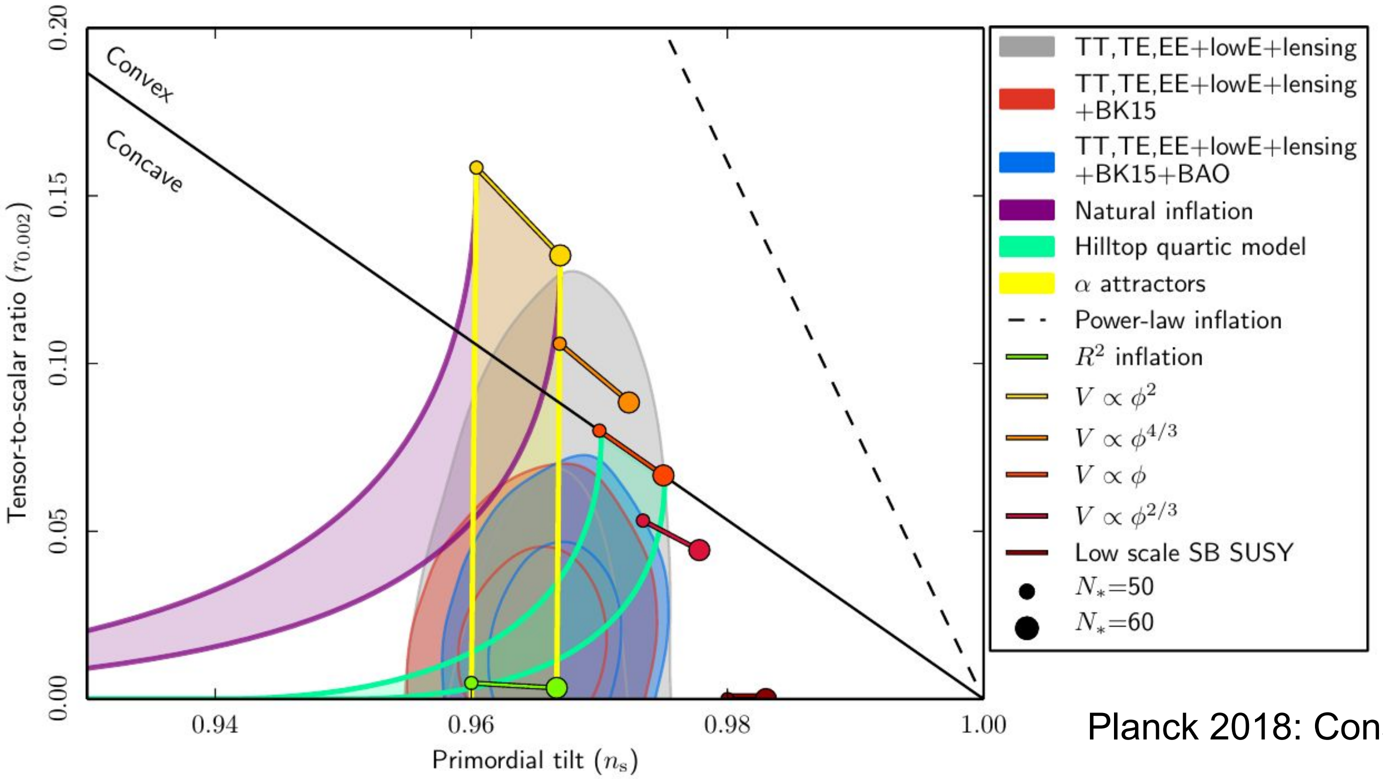


10.1088/0004-637X/779/1/86

http://www.preposterousuniverse.com/blog/2014/03/16/bicep2-updates/

The Institute of Mathematical Sciences

Cosmic Microwave Background (Planck)



Planck 2018: Constraints on inflation

Inflation models and Planck

Inflationary model	Potential $V(\phi)$	Parameter range	$\Delta\chi^2$	$\ln B$
$R + R^2/(6M^2)$	$\Lambda^4\left(1 - e^{-\sqrt{2/3}\phi/M_{\rm Pl}}\right)^2$	...	...	...
Power-law potential	$\lambda M_{\rm Pl}^{10/3}\phi^{2/3}$	...	4.0	−4.6
Power-law potential	$\lambda M_{\rm Pl}^3\phi$	...	6.8	−3.9
Power-law potential	$\lambda M_{\rm Pl}^{8/3}\phi^{4/3}$	...	12.0	−6.4
Power-law potential	$\lambda M_{\rm Pl}^2\phi^2$	...	21.6	−11.5
Power-law potential	$\lambda M_{\rm Pl}\phi^3$	...	44.7	−13.2
Power-law potential	$\lambda\phi^4$	...	75.3	−56.0
Non-minimal coupling	$\lambda^4\phi^4 + \xi\phi^2 R/2$	$-4 < \log_{10}\xi < 4$	0.4	−2.4
Natural inflation	$\Lambda^4\left[1 + \cos(\phi/f)\right]$	$0.3 < \log_{10}(f/M_{\rm Pl}) < 2.5$	9.9	−6.6
Hilltop quadratic model	$\Lambda^4\left(1 - \phi^2/\mu_2^2 + \ldots\right)$	$0.3 < \log_{10}(\mu_2/M_{\rm Pl}) < 4.85$	1.3	−2.0
Hilltop quartic model	$\Lambda^4\left(1 - \phi^4/\mu_4^4 + \ldots\right)$	$-2 < \log_{10}(\mu_4/M_{\rm Pl}) < 2$	−0.3	−1.4
D-brane inflation ($p = 2$)	$\Lambda^4\left(1 - \mu_{\rm D2}^2/\phi^p + \ldots\right)$	$-6 < \log_{10}(\mu_{\rm D2}/M_{\rm Pl}) < 0.3$	−2.0	0.6
D-brane inflation ($p = 4$)	$\Lambda^4\left(1 - \mu_{\rm D4}^4/\phi^p + \ldots\right)$	$-6 < \log_{10}(\mu_{\rm D4}/M_{\rm Pl}) < 0.3$	−3.5	−0.4
Potential with exponential tails	$\Lambda^4\left[1 - \exp(-q\phi/M_{\rm Pl}) + \ldots\right]$	$-3 < \log_{10} q < 3$	−0.4	−1.0
Spontaneously broken SUSY	$\Lambda^4\left[1 + \alpha_h \log(\phi/M_{\rm Pl}) + \ldots\right]$	$-2.5 < \log_{10}\alpha_h < 1$	6.7	−6.8
E-model ($n = 1$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_1^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$	$-2 < \log_{10}\alpha_1^{\rm E} < 4$	0.8	−0.3
E-model ($n = 2$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_2^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$	$-2 < \log_{10}\alpha_2^{\rm E} < 4$	0.8	−1.6
T-model ($m = 1$)	$\Lambda^4\tanh^{2m}\left[\phi\left(\sqrt{6\alpha_1^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$	$-2 < \log_{10}\alpha_1^{\rm T} < 4$	−0.1	−1.2
T-model ($m = 2$)	$\Lambda^4\tanh^{2m}\left[\phi\left(\sqrt{6\alpha_2^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$	$-2 < \log_{10}\alpha_2^{\rm T} < 4$	0.8	−0.6

Planck 2018: Constraints on inflation

BINGO

BIspectra and Non-Gaussianity Operator: https://github.com/dkhaz/bingo

This code can be used to calculate both power spectrum and bispectra for any single field canonical model of inflation

For our purposes we will just focus on the power spectrum part

Therefore, it is simpler to work with the ipynb

https://gitlab.com/dhirajhazra/simple-codes-in-cosmology/-/blob/master/Inflation-Primordial-Perturbation.ipynb