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Feynman Paths and the Wave function of Space-time

Einstein tensor determines the Euclidean amplitudes for a causal domain to emerge

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Feynman's path integral prescription for wave function of a system can be used to obtain the ground state wave function in terms of a suitably defined Euclidean path integral. I apply this prescription to obtain the ground state wave function of a Lorentzian spacetime $(M, \boldsymbol{g})$. This is done using timelike geodesics $\boldsymbol{u}$ emanating from an arbitrary event p. These not only define the casual domain of p, but also provide a natural definition of the Euclidean regime $(M, \boldsymbol{g}_{\rm E}, \boldsymbol{u})$ of $(M, \boldsymbol{g})$. I compute the leading order Einstein-Hilbert action $I_{\rm E} = I[\boldsymbol{g}_{\rm E}]$, and show that it is proportional to the Einstein tensor. Putting all this together finally yields a wave function Ψ for a Lorentzian spacetime $(M, \boldsymbol{g})$, whose interpretation is discussed.

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Feynman path integral and the Wave function

The path integral formulation of quantum mechanics assigns a specific weightage, $A[H]=e^{S[H]}$, to each possible classical history H of a given system, with $S[H]$ being the action. The key role is then played by the sum-over-histories of $A[H]$. This same sum, when supplied with suitable boundary conditions, also yields the wave-function of the system, as discussed in the original paper by Feynman [1]. The *euclidean* version of this prescription is then easily shown to yield the *ground state wave function* of the system. (See **Fig. 1.**)

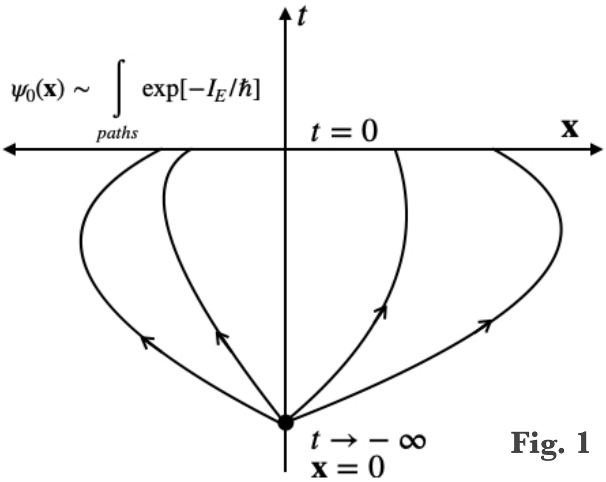

Fig. 1

In what surely must count as an audacious attempt, Stephen Hawking and Jim Hartle (HH) [2] sought to apply this formulation to the entire universe, hoping to use path integrals to define a wave-function for the universe. Their prescription goes in following steps (interspersed, of course, with rigorous maths that yields quantitative results):

1. Describe the universe by the Einstein-Hilbert action: $S[g] = \int L[g] = \int_V \text{RicSc}[g] + 2\int_{\partial V} K.$
2. Euclideanize the action by Wick rotation.
3. Evaluate the dominant saddle points of the path integral of $A[H]$ with some prescribed boundary conditions. Hence obtain the "wave function of the universe" Ψ.

The prescription is elegant and simple. Let me briefly explain the steps above: (1) is a fair enough input, although generalisation of HH analysis for higher curvature actions should be straightforward. (2) is motivated by its success in quantum field theories without gravity, where it works extremely well. *The issues with (2) for the case of gravity is what will largely form the backbone of this essay*. Finally, (3) is essentially the standard result for path integrals applied to the universe. The only contentious point here is the choice of boundary conditions. This was spelled out by HH, and goes by the name of the *no-boundary proposal*. We will not discuss this here, but instead get back to the issues with (2). Amongst the many issues one must deal with in proceeding to understand quantum gravity via path integrals, the most important is the illdefined prescription of Wick rotation $t \to \pm it$, in which one analytically continues the time coordinate in the hope that it will make the path amplitudes, and hence the sum over paths, well behaved. The prescription works in spacetimes possessing a static timelike Killing field, but otherwise, it is simply useless. What works so well in other gauge theories simply does not work for gravity. Many researches have discussed in depth the issues of Euclidean quantum gravity, and its implications for quantum cosmology, but the results, and the associated plethora of prescriptions for analytic continuations, muddy the whole framework, making its basic postulates unconvincing and it's utility unclear. Last, but not the least, of the issues with conventional Wick rotation is that it generically yields a *complex* metric, with no proper interpretation.

However, there does exists a nice, covariant, alternative to Wick rotation, provided by the class of metrics $g_{ab}^{(E)} = g_{ab} + \Theta(x)\, u_a u_b$, where $\Theta(x)$ is a scalar field whose gradient is parallel to the unit timeline vector field u_a; this function can be chosen in an appropriate manner; see **Fig. 2** for a typical profile, for which it is easily shown that $g_{ab}^{(E)}$ describe a class of metrics that start in the Euclidean phase ($\Theta < -1$), and make a (singular) transition to the Lorentzian phase ($\Theta > -1$) at some length scale, say ℓ_0. This length scale, which one expects to characterise the small scale structure of spacetime, is *a priori* independent of the Planck length, 10^{-33} cm. However, to avoid clutter, I will set it as equal to the Planck scale.

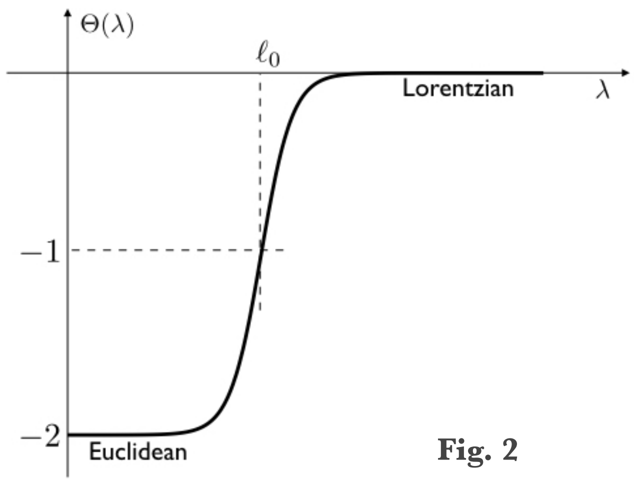

Fig. 2

Causal curves and Spacetime geometry

Having thus identified a class of metrics that have a well defined Euclidean regime, ask:

Q1. *How does one apply this formalism to the small scale structure of an arbitrary spacetime? In particular,*

Q1.1. *What u^a must one chose?*

Q2. *What becomes of the Einstein-Hilbert action in the Euclidean regime?*

Q3. *Which of the well known tensors: Ricci scalar R, Ricci tensor R_{ab}, Einstein tensor G_{ab}, or the curvature tensor R^{ab}_{cd}, determines the leading order Euclidean path amplitudes.*

For answers, first zoom in to an arbitrary neighbourhood of an arbitrary spacetime event p_0. Things in this *mesoscopic* domain will look somewhat like **Fig. 3**. Depicted here are timelike

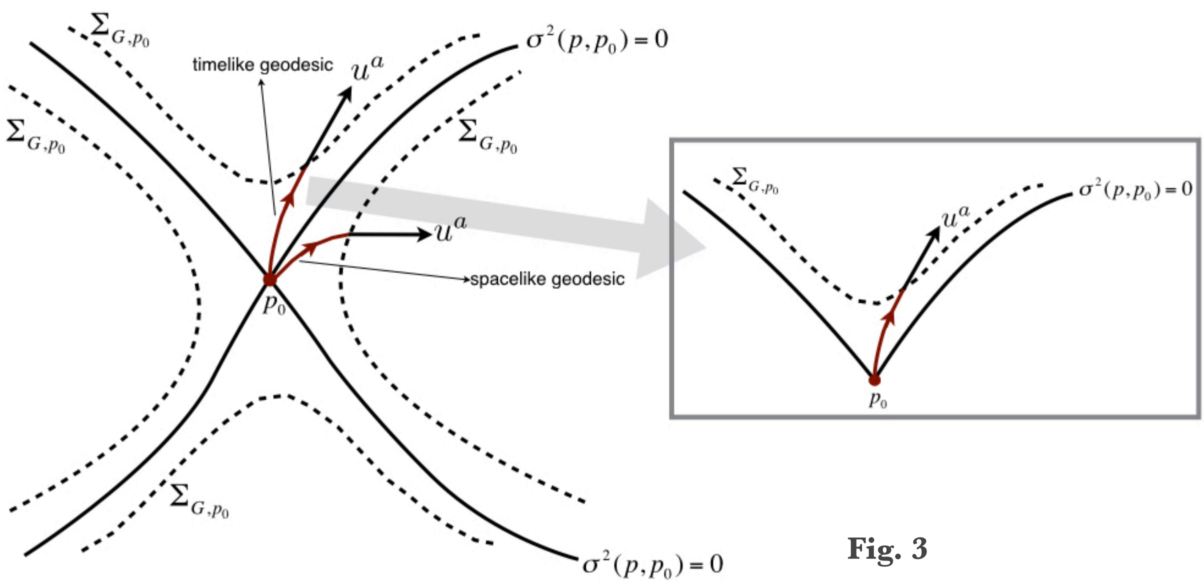

Fig. 3

geodesics and null cones anchored at p_0. Consider now all time like geodesics emanating from p_0, and set the affine parameter λ to zero at p_0. Now, it is simple to show that the congruence in consideration is *hyper surface orthogonal,* and hence one can move an affine distance $\lambda = \ell_0$ along each member of it, to construct a hyper surface on which our function Θ would take the value -1. This construction takes care of Q1 above. It also yields all the ingredients to answer Q2; we simply evaluate the Ricci scalar $R[g^{(E)}_{ab}]$ corresponding to $g^{(E)}_{ab}$. This requires exact expressions for various geometric quantities associated with the foliation that we have set up, such as its intrinsic and extrinsic curvature. These can be derived as covariant Taylor expansions in λ. The lowest order term will suffice for our considerations, since that also happens to yield the leading curvature dependence. Higher orders in λ will bring in higher curvature terms in the action, which I will ignore here.

The Euclidean Einstein-Hilbert action

After a long computation [3], we finally get an answer to Q3 that is, at the same time, unexpected and extremely intriguing:

$$\frac{I_E}{\hbar} \approx \frac{2}{D} \ell_0^2 G_{ab}(p_0)\, \tau^{ab}$$

where τ^{ab} represent the average of $u^a u^b$ over the unit $(D-1)$ dimensional hyperbolic space. We note, in particular, that

$$\textit{If } G_{ab}u^a u^b > 0 \textit{ for all time like vectors } u^a, \textit{ so is } I_E.$$

For the cognoscenti, the significance of the above result can not be overemphasised. It hints at a connection between the so-called *positive action conjecture* in Euclidean gravity, with (the geometrical version of) the *weak energy condition* in classical general relativity. [1]

Wave function of spacetime and Einstein tensor

One now uses the Euclidean action, and interprets the ground state wave function

$$\Psi_{p_0}\left[\Sigma_{G,p_0}\right] \sim e^{-(2/D)\ell_0^2 G_{ab}\tau^{ab}}$$

as a function whose modulus squared would give the probability for a space like hypersurface Σ_{G,p_0} to emerge at a constant geodesic distance ℓ_0 from a spacetime event p_0. Applied to all spacetime events, one then defines the total wave function by $\Psi = \prod_{p_0} \Psi_{p_0}$. *What would be the*

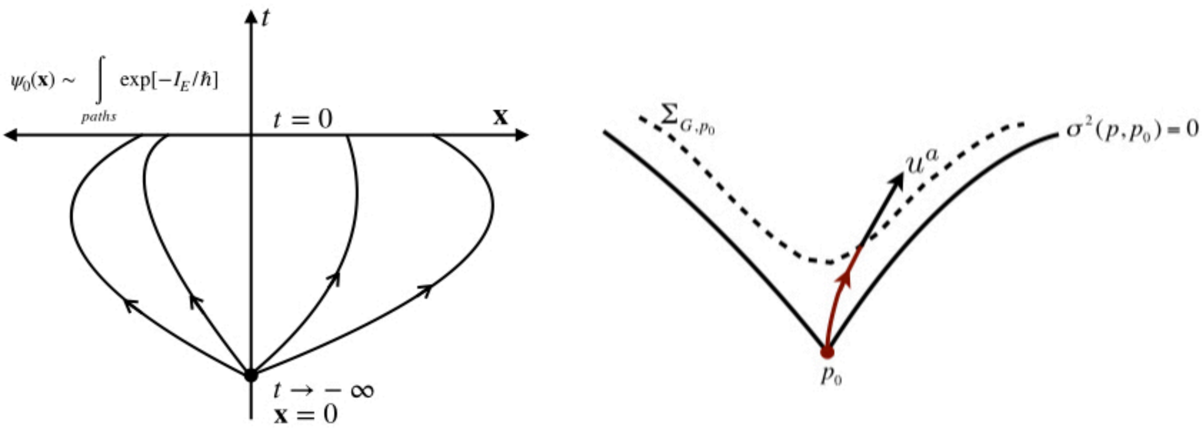

Fig. 4 Comparison between Feynman's definition of ground state wave function for point particle paths, and the set-up described here involving geodesic timelike paths. The construction suggested here therefore seems to be more than similarities of prescription; it suggests that timelike curves indeed do play a key role in emergence of spacetime in a quantum theory.

[1] There is already a connection between positive energy theorem in (*D*+*1*) dimensions and positive action conjecture in *D* dimensions - former implies the latter.

meaning of this wave function? This is not completely clear, but presumably it must have some interpretation in terms of emergence of a spacetime with a given Lorentzian metric g_{ab}.

There exists a very close relationship between our set-up and the original one by Feynman, although we are not doing quantum dynamics of point particle. This is depicted and explained in **Fig. 4**. These observations might not only help provide a rigorous justification for our boundary conditions and interpretation of the wave function, but might even help in evaluating and understanding better the structure of the gravitational path integral

$$Z = \int \mathscr{D}g_{ab}\, \mathscr{D}u^a \; \exp i I[g_{ab}^{(E)}]$$

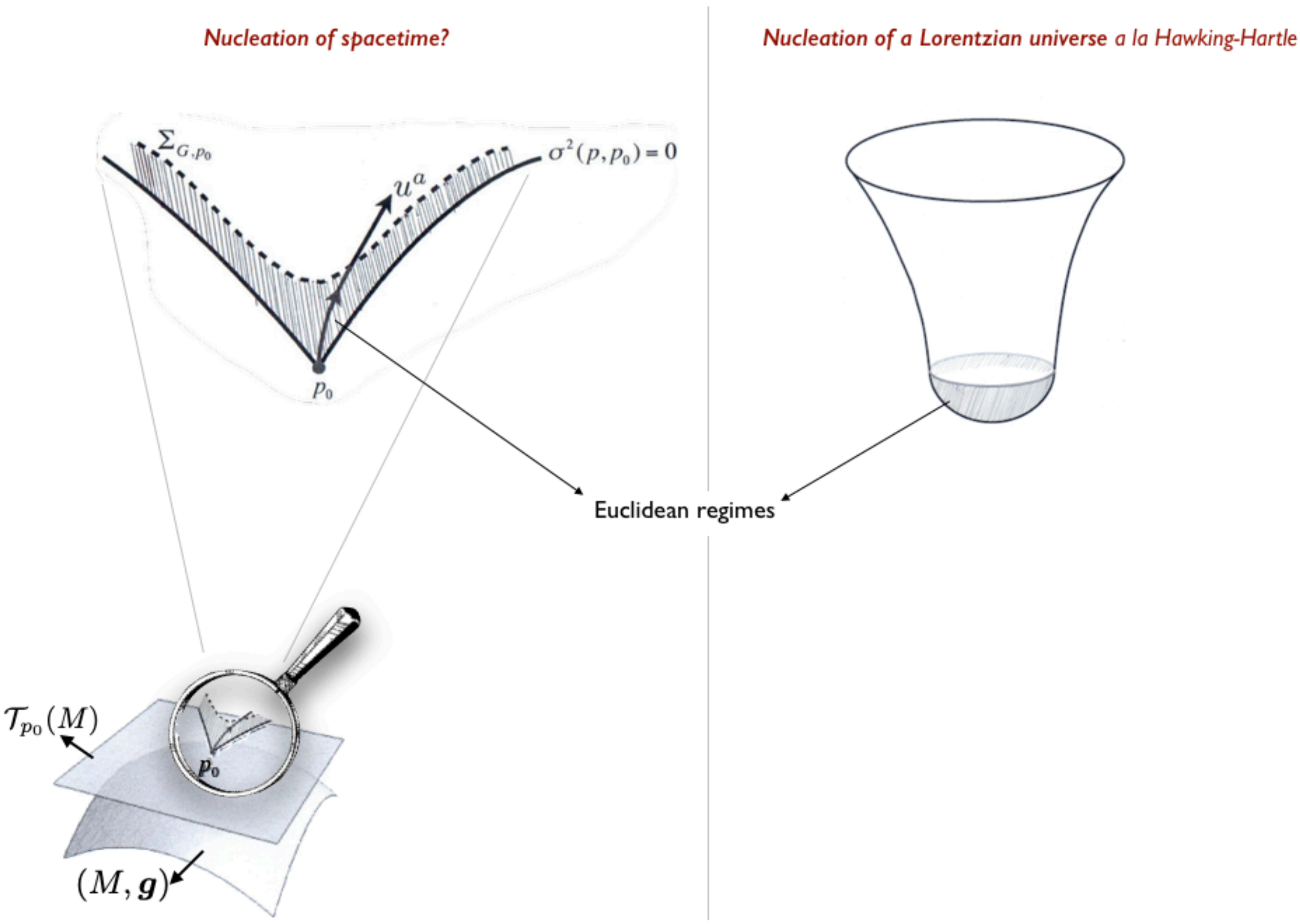

Fig. 5 *Left*: Emergence of spacetime can be described in terms of emergence of a causal future (or past) of an event p_0. This, in turn, is described in terms of a wave function that is determined by the Euclidean action constructed from all timelike geodesics emanating from p_0. *Right*: Nucleation of a Lorentzian universe from a Euclidean phase *a la* Hawking and Hartle. The figure illustrates conceptual similarities of the two ideas. The key difference is the observer dependent euclideanisation applied to arbitrary spacetime here.

This remains a daunting task. Daunting, but not impossible! One can indeed extract quantitative results by plugging in the action and attempting to evaluate Z in some suitable approximation. The set-up we have described is essentially rooted in the question about what would happen if spacetime turns euclidean at the smallest of scales. Very often, following the path set by HH, it has been considered that spacetime near *singularities* might be euclidean. What we are saying here is similar in spirit, but different in almost every other aspect! (see **Fig. 5**). Our results depend, of course, on the specific "proposal" for Euclideanization which, being so, can not be *derived* any more that one can derive Wick rotation. However, IF one accepts the proposal given here, and the resultant Euclidean metric it yields, rest of the conclusions follow rigorously. Specifically, the result establishes a very direct relation between a quantum wave function Ψ and the Einstein tensor G_{ab}, a relation that arises in no trivial manner from specific mathematical details of the geodesic structure of spacetime (see [3] for details). A final step would be to incorporate the effects due to vacuum fluctuations [4], and account for yet another feature of small scale structure of spacetime - the existence of a lower bound to spacetime intervals [5]. These are under investigation.

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