Holography, quantum gravity and confinement

Viqar Husain[1,2*] and Dawood Kothawala[2†]
[1] *Perimeter Institute for Theoretical Physics, 31 Caroline St. N. Waterloo ON, Canada N2L 2Y5 and*
[2] *Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3*
(Dated: March 28, 2012)

In the holographic dictionary between gauge theory in four dimensions and gravity in five dimensions, there is an encoding in the bulk geometry of the phases of the gauge theory. If the correspondence holds at all scales, it is natural to expect that gauge theory contains information about quantum gravity in one higher dimension. We argue that the confining phase of gauge theory has a correspondence with singularity avoidance in quantum gravity. This comes from the observation that confinement appears to be generically associated with repulsion deep in the bulk on the gravity side, which in turn is a consequence of the violation of energy conditions in quantum gravity that lead to singularity avoidance.

Essay written for the Gravity Research Foundation 2012 Awards for Essays on Gravitation

I. INTRODUCTION

Among the various intriguing insights that have come from black hole physics is the idea of holography [1, 2]. Broadly speaking this is the encoding of physical degrees of freedom of a theory on the boundary of the region in which it is defined. A concrete example of such a duality between bulk and boundary theories is the AdS/CFT correspondence [3], where a string theory on an asymptotically AdS spacetime is conjectured to be equivalent to a certain superconformal Yang-Mills theory in one lower spacetime dimension.

In its strong version, the AdS/CFT conjecture is an *exact duality between two quantum theories* (see eg. [4]). If true, this would mean that certain supersymmetric gauge theories contain information about quantum gravity in one higher dimension. This raises many physical and structural questions about both theories that have yet to be satisfactorily addressed. Some of these are the following. How is gravitational collapse, Hawking radiation and singularity avoidance in quantum gravity interpreted in the gauge theory? What is the role of diffeomorphism invariance of gravity in the gauge theory? How are the phases of gauge theory manifested in the gravity theory?

In this essay we attempt an understanding of the last of these questions. Our argument is based on a specific dictionary entry in the AdS/CFT correspondence, namely a calculation on the gravity side of the expectation value of the Wilson loop observable in gauge theory. We argue that *the confinement phase of gauge theory might be intimately connected with singularity avoidance in quantum gravity.*

We begin by summarizing what we expect of a quantum theory of gravity, and highlight those features of the AdS/CFT correspondence that we use to develop the argument. The problem of quantum gravity may be approached from several directions. The major divisions concern the starting point, namely what theory to quantize, and following this, whether one should follow a perturbative or non-perturbative approach, and whether to use Dirac, reduced phase space, or covariant quantization. Regardless of the approach however, there appears to be agreement on the problems that are to be solved by a theory of quantum gravity:

- Resolution of classical curvature singularities.
- Understanding gravitational collapse, Hawking radiation, and the information loss problem.
- The problem of time and background independence.

In the AdS/CFT correspondence and its various derivatives pertaining to QCD and condensed matter physics [5], classical or semi-classical calculations on the gravity side play a prominent role in addressing fully quantum problems in the boundary theory. All such dictionary "mini-conjectures" have a feature known as the UV/IR relation [6]. This is the statement that radial distance in the bulk is related to size on the boundary. Specifically, if r is a radial bulk coordinate, then small r corresponds to low energy boundary phenomena; conversely, large r corresponds to high

*Electronic address: vhusain@unb.ca
†Electronic address: dkothawa@unb.ca

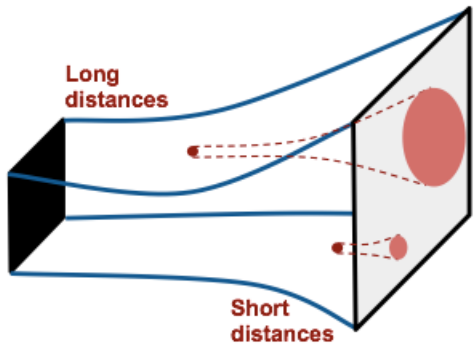

FIG. 1: A depiction of the UV/IR relation (from [5])

energy on the boundary. This UV/IR correspondence is expected to be a generic feature of holographic dualities. The features of holography we assume are the following:

- There is an precise correspondence between a quantum theory of gravity and a quantum gauge theory in one lower dimension.
- The renormalization group flow of the boundary theory is encoded in the equations of the bulk gravitational theory.

II. WILSON LOOP, UV/IR, AND CONFINEMENT

With the above ingredients in hand let us consider the Wilson loop calculation in the AdS/CFT correspondence [7]. This dictionary entry takes as input an asymptotically AdS bulk spacetime and produces a quark-antiquark ($q\bar{q}$) potential in the boundary theory. For instance, the planar global AdS metric gives the Coulomb potential and the Schwarzschild-AdS metric gives a screened Coulomb potential (i.e., there is a finite separation of the $q\bar{q}$ pair where the potential is zero).

Of particular interest for our purpose are those bulk spacetimes that give a confining potential [8, 9]. Since confinement is a low energy phenomenon, the UV/IR aspect of holography indicates that it must correspond to deep bulk physics. This is what we now explore.

The prescription for computing the Wilson loop expectation value is [7, 8]

$$\langle W_\gamma \rangle = \int D s_\gamma \exp\left(-S_{\rm NG}\{s_\gamma\}\right) \tag{1}$$

where γ is the boundary loop, s_γ is a bulk world sheet with γ as its boundary, and $S_{\rm NG}\{s_\gamma\}$ is the Nambu-Goto string action. The r.h.s of this equation is typically approximated at the saddle point, giving

$$\langle W_\gamma \rangle = \exp\left[-\left(S_{\rm NG}\{s_{\gamma(\rm min)}\} - S_{\rm reg}\right)\right] \tag{2}$$

where $s_{\gamma(\rm min)}$ is the minimal worldsheet area with the loop γ as its boundary. This is obtained by extremizing the classical string action $S_{\rm NG}$. $S_{\rm reg}$ is the action of two rectangular worldsheets extending into the bulk that have the two timelike edges of the Wilson loop as their boundaries. This subtraction corresponds to the energy of the two free quarks moving on the boundary. (The suffix "reg" signifies the fact that in an AAdS spacetime, this term removes the divergence coming from the first term.)

Let us consider the planar metric

$$\mathrm{d}s^2 = \frac{r^2}{\ell^2}\left[-f(r)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] + \frac{\ell^2}{r^2}\frac{\mathrm{d}r^2}{g(r)} \tag{3}$$

where the functions $f(r)$ and $g(r)$ are such that the spacetime is asymptotically AdS_5. The procedure is to analyze the minimal worldsheet which is bounded by a rectangular loop in a timelike plane on the boundary (which we take

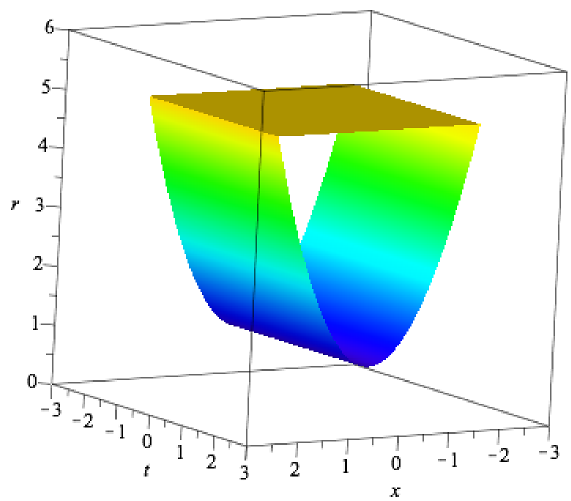

FIG. 2: Illustration of the rectangular Wilson loop; here $r_m \sim 1$ is the minimum radius in the bulk to which the world sheet descends.

to lie in the $t-x$ plane for definiteness, with sides T and L respectively), and obtain parametric expressions for $q\bar{q}$ potential V as well their boundary separation L in terms of minimal worldsheet radius r_m; a parametric plot of $V(r_m)$ vs. $L(r_m)$ then gives the desired $V-L$ curve. The worldsheet (Fig. 2) is described by a single function $r(x)$, and has the induced metric

$$h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\frac{r^2 f(r)}{\ell^2}\mathrm{d}t^2 + \left(\frac{r^2}{\ell^2} + \frac{\ell^2}{r^2 g(r)}\left(\frac{\partial r}{\partial x}\right)^2\right)\mathrm{d}x^2 \tag{4}$$

Therefore, the Nambu-Goto action becomes (for clarity, we will restore the dimensionful constants such as string tension etc. only at the end)

$$S_{NG} = T\int \mathrm{d}x\sqrt{\frac{r^4 f(r)}{\ell^4} + \frac{f(r)}{g(r)}r'^2} \tag{5}$$

where $r' = \partial r/\partial x$ and the integral over t is trivial due to t independence of the integrand. The minimal surface is obtained by noting that the integrand does not depend on x explicitly, so the "Hamiltonian" corresponding to x translations is a constant. This constant can be fixed by evaluating the Hamiltonian at the minimal radius, r_m, of the worldsheet, where $r' = 0$. This gives

$$r' = \frac{r^2}{\ell^2}\sqrt{\frac{r^4}{r_m^4}\frac{fg}{f_m} - g} \tag{6}$$

where $f_m = f(r_m)$. Integrating this with the condition $x(\infty) = \pm L/2$ (symmetry then dictates that $x(r_m) = 0$), we obtain

$$L(r_m) = \frac{2\ell^2}{r_m}\int_1^\infty \frac{\mathrm{d}y}{y^2}\frac{1}{\sqrt{g(y;r_m)}}\frac{1}{\sqrt{\frac{f(y;r_m)}{f_m(r_m)}y^4 - 1}} \tag{7}$$

where $y = r/r_m$, and $f_m(r_m) = f(1; r_m)$. This equation gives the $q\bar{q}$ separation L as a function of the minimum radial coordinate value r_m of the world sheet.

The interaction potential $V(r_m)$ is obtained using the prescription

$$S_{NG} = TV(L) \tag{8}$$

with the action computed for the minimal surface. Using $r'(x)$ from Eqn. (6) gives

$$S_{\rm NG} = \frac{2Tr_m}{f_m(r_m)} \int_1^\infty {\rm d}y \frac{y^2 f(y;r_m)}{\sqrt{g(y;r_m)}\sqrt{\frac{f(y;r_m)}{f_m(r_m)}y^4 - 1}}$$

To obtain the interaction potential from this, we first need to regularize this expression by subtracting the contribution coming from the edges of the loop along the t direction, which has the interpretation of energy of free quarks. Since these two worldsheets lie in the $t - r$ plane, the corresponding Nambu-Goto action is

$$S_{\rm reg} = 2T \int_{\Lambda_c}^\infty {\rm d}r \sqrt{\frac{f}{g}} = 2Tr_m \int_{\Lambda_c/r_m}^\infty {\rm d}y \sqrt{\frac{f}{g}} \tag{9}$$

where $r = \Lambda_c$ is some cut-off radius up to which these free-quark worldsheets extend in the bulk. In pure AdS bulk, one can take $\Lambda_c = 0$, whereas for a black hole in AdS, $\Lambda_c = r_0$, the horizon radius. The potential $V(r_m)$ is therefore

$$V(r_m) \;=\; \frac{1}{T}\left(S_{\rm NG} - S_{\rm reg}\right) \tag{10}$$

Putting everything together gives

$$\begin{aligned} \frac{V(r_m)}{\hbar c r_0/\alpha'} &= \frac{2r_m}{r_0}\left[\int_1^\infty {\rm d}y \sqrt{\frac{f(y;r_m)}{g(y;r_m)}}\left(\sqrt{\frac{f(y;r_m)}{f_m(r_m)}}\frac{y^2}{\sqrt{\frac{f(y;r_m)}{f_m(r_m)}y^4 - 1}} - 1\right) - \int_{\Lambda_c/r_m}^1 {\rm d}y \sqrt{\frac{f(y;r_m)}{g(y;r_m)}}\right] \\ \frac{L(r_m)}{\ell^2/r_0} &= \frac{2r_0}{r_m}\int_1^\infty \frac{{\rm d}y}{y^2}\frac{1}{\sqrt{g(y;r_m)}}\frac{1}{\sqrt{\frac{f(y;r_m)}{f_m(r_m)}y^4 - 1}}, \end{aligned} \tag{11}$$

where the string length scale, $\ell_s = \sqrt{\alpha'}$ is indicated. These formulas contain all the information we need to argue for a connection between singularity avoidance and confinement, as we now see.

This calculation has been done for the global AdS and the AdS-Schwarzschild metrics [7]. As expected for black hole solutions, the latter metric has positive energy, a horizon and a curvature singularity at $r = 0$. Now we know that singularity avoidance necessarily requires a violation of energy conditions. Therefore we ask *what the Wilson loop calculation yields for spacetimes with negative energy, whether or not there is a curvature singularity.*

For this purpose we consider two spacetimes. The first is a regulated negative mass Schwarzschild metric given by (3) with the functions

$$f(r) \;=\; 1 + \frac{r_0^4}{r^4}, \tag{12}$$

$$g(r) \;=\; 1 + \frac{r_0^4}{r^4}\left[1 - \exp\left(-\frac{r^4}{\lambda^4}\right)\right]. \tag{13}$$

For this choice the Kretschmann scalar is finite everywhere, in particular at $r = 0$ the behaviour is

$$R_{abcd}R^{abcd} \;\xrightarrow{r=0}\; \frac{40}{\ell^4}\left[1 + \left(\frac{r_0}{\lambda}\right)^4\right]^2. \tag{14}$$

The second is the the AdS soliton [10] with metric

$$ {\rm d}s^2 = \frac{r^2}{\ell^2}\left[-{\rm d}t^2 + {\rm d}x^2 + f(r){\rm d}y^2 + {\rm d}z^2\right] + \frac{\ell^2}{r^2}\frac{{\rm d}r^2}{g(r)} \tag{15}$$

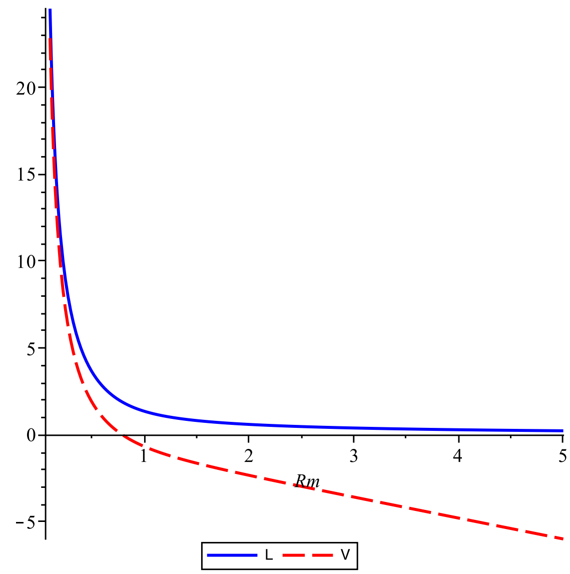

FIG. 3: $q\bar{q}$ potential $V(r_m)$ and separation $L(r_m)$ for the negative mass AdS-Schwarzschild geometry. This shows a UV/IR relation: large L on the boundary correspond to small r_m and vice versa.

with $f = g = 1 - r^4/r_0^4$. This solution has a negative energy, which unlike the negative mass Schwarzschild case, is bounded below. The worldsheet we consider has the same boundary loop in the $t - x$ plane, but its induced metric is now

$$\mathrm{d}s^2 = -\frac{r^2}{\ell^2}\mathrm{d}t^2 + \left(-\frac{r^2}{\ell^2} \; + \; r'(x)^2 \frac{\ell^2}{r^2 g(r)}\right)\mathrm{d}x^2 \tag{16}$$

Therefore the potential $V(L)$ for this case is obtained by simply setting $f = 1$ and $g = 1 - r_0^4/r^4$ in the formulas (11).

Figure 3 shows graphs of $V(r_m)$ and $L(r_m)$ for the regulated negative mass Schwarzschild black hole. It is apparent that the curves exhibit the type of UV/IR duality shown in Fig. 1: small r_m (deep bulk) corresponds to large L. Conversely, large r_m corresponds to small L. Expanding the integrands in a Taylor series at $r_m = 0$ for the regulated negative mass Schwarzschild case gives

$$V = \left(\frac{r_0}{\ell}\right)^2 L - \frac{2\lambda^2 r_0^2}{\sqrt{r_0^4 + \lambda^4}}\frac{1}{\Lambda_c} + O(r_m^3, \Lambda_c^3) \tag{17}$$

This demonstrates the confining behaviour, and gives the slope and intercept of the $V(L)$ line; the expansion that gives this result obviously cannot be done for the positive mass case due to the square roots in the integrands. The regulator and cutoff lengths λ and Λ_c determine the intercept.

Figure 4 gives the potentials $V(L)$ for the negative mass AdS-Schwarzschild geometry and the AdS soliton. *It is apparent that the confining regions for large L correspond to negative energy repulsion in deep bulk region* (ie. small r_m). The termination of the AdS soliton line in Fig. 4 may be interpreted as a correspondence with the breaking of the streched $q\bar{q}$ string; this does not happen for the negative mass Schwarzschild case because, unlike the AdS soliton, it has no energy lower bound.

III. RELATION TO SINGULARITY AVOIDANCE IN QUANTUM GRAVITY

We have seen two explicit examples showing that gravitational repulsion deep in bulk, due to negative energy, gives confinement in the boundary gauge theory. That such behaviour is generic may be seen with a physical argument: string world sheets that extend deep in the AdS bulk feel a repulsion due to negative energy, which in turn causes the boundary loop to be stretched [11]. This is apparently what signals confinement in the boundary gauge theory.

Now we also know that quantum corrected gravitational field equations in the bulk should have regular solutions that avoid curvature singularities. But this can only happen if there are regions where at least the weak energy condition,

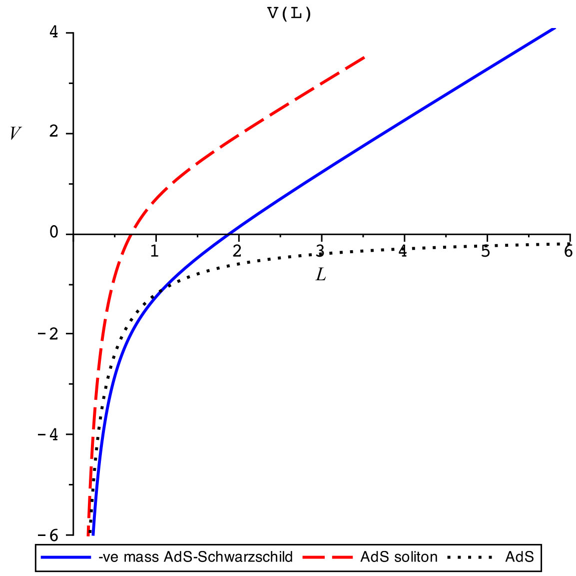

FIG. 4: $q\bar{q}$ potential $V(L)$ from AdS, AdS soliton and negative mass AdS-Schwarzschild geometries; the latter two show a Coulomb to confinement transition. The AdS soliton line terminates at $r_m = r_0$, which corresponds to its (negative) low energy bound.

and possibly also the dominant one is violated. Furthermore, invoking the UV/IR correspondence, the radial gravity equations are also the RG flow equations of the boundary theory. Combining these observations suggests the sequence of links

$$\textbf{singularity avoidance in QG} \longleftrightarrow \textbf{negative energy} \longleftrightarrow \textbf{confinement in gauge theory}.$$

If the duality between gauge theory and gravity is an exact correspondence between the respective quantum theories, then expected features of quantum gravity must have manifestations in gauge theory. Using the idea behind the Wilson loop calculation in the AdS/CFT correspondence, we have argued that *confinement in gauge theory is intimately and generically connected with singularity avoidance in quantum gravity, while maintaining the UV/IR aspect of holography.* It would be very interesting to see if other calculations support this hypothesis.

Acknowledgements We thank Jaume Gomis and Rob Myers for comments. This work was supported in part by the Natural Science and Engineering Research Council of Canada. The work of D.K. was also supported by an AARMS fellowship.

[1] G. 't Hooft (1993), essay dedicated to Abdus Salam, gr-qc/9310026.
[2] L. Susskind, J.Math.Phys. **36**, 6377 (1995), hep-th/9409089.
[3] J. M. Maldacena, Adv.Theor.Math.Phys. **2**, 231 (1998), hep-th/9711200.
[4] J. de Boer, pp. 512–527 (2002).
[5] S. A. Hartnoll (2011), 1106.4324.
[6] L. Susskind and E. Witten (1998), hep-th/9805114.
[7] J. M. Maldacena, Phys.Rev.Lett. **80**, 4859 (1998), hep-th/9803002.
[8] E. Witten, Adv.Theor.Math.Phys. **2**, 505 (1998), hep-th/9803131.
[9] Y. Kinar, E. Schreiber, and J. Sonnenschein, Nucl.Phys. **B566**, 103 (2000), hep-th/9811192.
[10] G. T. Horowitz and R. C. Myers, Phys.Rev. **D59**, 026005 (1998), hep-th/9808079.
[11] We thank Gary Horowitz for a clarifying comment regarding this.