Thermal Entropy and Spacetime Curvature

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Abstract

Spacetime *curvature* will generically perturb the *energy eigenvalues* of a system – a fact which can lead to interesting effects particularly in *thermal* properties of the system. I discuss a simple example where such an interplay between curvature and quantum mechanics indeed leads to a specific, sub-dominant term in thermal properties of a freely falling system at high temperature, characterized by a fundamental dimensionless quantity: $\Delta = R_{00}(\hbar c/kT)^2$. Intriguingly, the Δ-terms exhibit several features familiar from *black hole thermodynamics*. This suggests that at least some features of black hole thermodynamics might be understood better by studying how *ordinary* thermal systems behave in presence of spacetime curvature.

. . . maybe that is a way of defining simplicity. Perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing. – Feynman (1965)

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Gravity is spacetime curvature, and hence the most ubiquitous of all interactions: it couples to *everything*, and operates *unshielded* with an *infinite range*. Statistical mechanics in presence of such interactions exhibits several peculiar features, something which astrophysicists have known since a long time [1]. It just so happened that many of these peculiarities, such as negative specific heat, attracted attention only after they were encountered in the context of black holes. A black hole horizon magnifies quantum effects in its vicinity, revealing a gamut of exotic features, the most famous being its thermal attributes [2]. This "Gravity $\leftrightarrow$ Quantum $\leftrightarrow$ Thermodynamics" connection has been gaining increasing attention for over more than a decade now, due to it's potential relevance for our understanding of gravity, and perhaps spacetime itself, at a fundamental level.

However, what is often *not* appreciated is the fact that a proper understanding of black hole thermodynamics might also necessitate a proper understanding of statistical mechanics of thermal systems in presence of spacetime curvature. Moreover, since results from black hole thermodynamics are widely looked upon as important guideposts for candidate models of quantum gravity, such an understanding may also be important in addressing questions like:

- How much of quantum gravitational physics do we need to understand the thermodynamic aspects of black holes?

- To what extent are these aspects specific to black holes?

Take, for example, black hole entropy: $S_{BH} = 10^{66} \times \text{area[cm}^2]/4$. There are a large number of proposed explanations for it, and no unambiguous criteria to identify which one is most "fundamental". In such a situation, it is but natural to go to the basics and analyse how the laws that we *do* understand work in presence of gravity. For example, one can study ordinary matter entropy in presence of spacetime curvature, and see how (if at all) it transmutes into black hole entropy as the matter collapses across the event horizon. Although a tough ask, some preliminary attempts have been made towards the same [3].

In this essay, I wish to focus on the issue of matter entropy from a different point of view which could provide greater fundamental insight into thermodynamic aspects of gravity: *study of thermal systems in presence of spacetime curvature*. The motivation, however, remains the same: to probe the possibility that certain features of black hole thermodynamics might simply be features of standard thermodynamic systems when spacetime curvature is taken into account [4]. Studying such systems might therefore provide a glimpse into possible interplay of spacetime curvature and quantum mechanical behaviour of the system, and how these affect it's thermal properties. I will argue in this essay that some intriguing effects may indeed arise from such an interplay, based on a simple example of a box of ideal, non-relativistic gas at temperature $kT = \beta^{-1}$ falling freely in a curved spacetime. The curvature of spacetime perturbs energy eigenvalues of the system [1], yielding corrections to the

[1] Similar analysis was done for a Hydrogen atom by Parker [5].

partition function; amongst these is a very specific term which is independent of system details (box dimensions, mass of the particles etc), and is characterized by a dimensionless quantity:

$$\Delta = R_{\widehat{0}\widehat{0}} \left(\hbar c/kT\right)^2 \tag{1}$$

where $R_{\widehat{0}\widehat{0}} = R_{ab}u^a u^b$, u^a being the four-velocity of the center of the box. It is this term which will be our point of focus due to its universal form (it is independent of any system details other than temperature), and indeed, it leads to corrections which closely resemble the thermal features of black holes.

Let me now sketch the main steps leading to this and other related results. I focus on a local neighbourhood around a geodesic trajectory (about which the box will be constructed, see below), which can be described using Fermi Normal Coordinates (FNC) in which the metric takes the form [6] ($\mu, \nu, \ldots = 1, 2, \ldots$)

$$\begin{aligned} \mathrm{d}s^2 = \; & - \left[\left(1 + \frac{a_\mu y^\mu}{c^2}\right)^2 + R_{\widehat{0}\mu\widehat{0}\nu} y^\mu y^\nu\right] c^2 \mathrm{d}\tau^2 - \frac{4}{3} c R_{\widehat{0}\rho\mu\sigma} y^\rho y^\sigma \mathrm{d}y^\mu \mathrm{d}\tau \\ & + \left[\delta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma} y^\rho y^\sigma\right] \mathrm{d}y^\mu \mathrm{d}y^\nu \; + \; \text{cubic order terms} \end{aligned} \tag{2}$$

where $\boldsymbol{a} = \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u} = 0$ in our case. In these coordinates, $\boldsymbol{u} = \boldsymbol{\partial}_{\widehat{0}}$ (i.e., the original trajectory is simply $y^\mu = 0$), and I *define* the "box" as a confined region with flat "coordinate" faces, i.e., $y^1 \in [-L_1/2, +L_1/2]$ and similarly for y^2, y^3. The box is filled with an ideal gas at temperature β^{-1}.

The Hamiltonian for the constituent particles in FNC is $H = -c p_{\widehat{0}}$, which, using $p^2 = -m^2 c^2$, can be written as

$$H = \frac{g^{\widehat{0}\mu} p_\mu c}{g^{\widehat{0}\widehat{0}}} + \sqrt{\frac{g^{\mu\nu} p_\mu p_\nu c^2 + m^2 c^4}{-g^{\widehat{0}\widehat{0}}} + \left(\frac{g^{\widehat{0}\mu} p_\mu c}{g^{\widehat{0}\widehat{0}}}\right)^2} \tag{3}$$

More discussion on the physical significance of this Hamiltonian can be found in [7]. In the non-relativistic limit, after ignoring (second order) terms such as $\mathcal{R} y^2 \times (p/mc)^2$, it is easy to see that the $c \to \infty$ limit of the Hamiltonian is given by

$$H - mc^2 \approx \frac{p^2}{2m} + \frac{1}{2} m c^2 R_{\widehat{0}\mu\widehat{0}\nu} y^\mu y^\nu \tag{4}$$

In what follows, I will also ignore the time dependence carried by curvature components, a reasonable assumption if the time scale $\mathcal{R}/\dot{\mathcal{R}}$ is much larger compared to typical time scale associated with the gas; one expects this to be the case at high temperature.

The energy eigenvalues can be found using first order perturbation theory (i assume L_1, L_2 and L_3 to be incommensurable, so that non-degenerate perturbation theory can be used), which gives

$$E\left[\{n_i\}\right] \;=\; \underbrace{\frac{\hbar^2\pi^2}{2m}\sum_{i=1..3}\frac{n_i^2}{L_i^2}}_{\text{unperturbed eigenvalues}} \;+\; \underbrace{\frac{1}{24}mc^2\sum_{i=1..3}R_{\widehat{0}\widehat{i}\widehat{0}\widehat{i}}L_i^2\left(1-\frac{6}{\pi^2 n_i^2}\right)}_{\text{perturbation due to spacetime curvature}} \tag{5}$$

We now have the necessary input to evaluate corrections to the partition function. For N particles, assuming Boltzmann statistics, this is given by $Z = z^N/N!$ where z is the 1-particle partition function

$$z \;=\; \sum_{\{n_i\}} e^{-\beta E[\{n_i\}]} \tag{6}$$

which can be evaluated judiciously by approximating intermediate sums as integrals, and assuming $\lambda^3/V \ll 1$ where $\lambda = h/\sqrt{2\pi mkT}$ is the thermal de Broglie wavelength [2]. I skip the steps (which involve careful handling of divergent series [7]), and simply quote the final expression for the canonical partition function:

$$\boxed{\ln\left(\frac{Z}{Z_{\mathrm{F}}}\right) = -\frac{1}{2}Nc_1R_{\widehat{0}\widehat{0}}\Lambda^2 + N\sum_{i=1..3}\left(c_2-\frac{\pi}{12}\delta_i\right)\mathcal{R}_i\delta_i\Lambda^2} \tag{7}$$

where $\ln Z_{\mathrm{F}} = \ln(V^N\lambda^{-3N}/N!)$ is the flat space expression, and the following quantities have been introduced:

1. $\Lambda = \beta\hbar c$ – a length scale independent of box dimensions L_i and mass m (unlike λ)
2. $\mathcal{R}_i = R_{\widehat{0}\widehat{i}\widehat{0}\widehat{i}}$ and $\delta_i = L_i/\lambda$ (where $\delta_i \gg 1$ by assumption), and
3. c_1, c_2 are constants depending on second and first derivatives of the series

$$q(s) \;=\; \sum_{n=1}^{\infty}\frac{s\exp\left[-(\pi/4)s^2n^2\right]}{n^2} \tag{8}$$

Their numerical values can be shown to be $c_1 = 1/2$, $c_2 = \pi/12$ [7].

We can now obtain corrections to various thermodynamic quantities: $U_{\mathrm{corr}} = U - U_{\mathrm{F}}$ and $S_{\mathrm{corr}} = S - S_{\mathrm{F}}$, where $U_{\mathrm{F}} = 3N/2\beta$ and $S_{\mathrm{F}} = 3N/2 + N\ln\left(eV/N\lambda^3\right)$ are standard flat space

[2] It is easy to see that our assumptions of the gas being non-relativistic, and $L_i/\lambda \gg 1\ \forall i$, imply that the range of temperature over which the entire calculation is valid is: $\hbar^2/mL^2 \ll kT \ll mc^2$ (where $L = \min\{L_i\}$ and factors of 2π etc have been ignored). The ratio of upper to lower limits here is: $(L/\lambda_c)^2$, where $\lambda_c = \hbar/mc$ is the Compton wavelength. For ordinary systems, this provides a considerable range over which kT can vary while still satisfying the approximations made in the analysis.

expressions. Using standard definitions $U = -\partial_\beta \ln Z$ and $S = \ln Z + \beta U$ to evaluate $U_{\rm corr}$, $S_{\rm corr}$ and heat capacity at constant volume, $C_V = -\beta^2 \partial_\beta U = 3N/2 + C_{V\rm corr}$, we obtain

$$
\begin{aligned}
2S_{\rm corr}/N &= \underbrace{+c_1 R_{\widehat{0}\widehat{0}} \Lambda^2}_{2s_\Delta/N} - c_2 \Lambda^2 \sum_{i=1..3} \mathcal{R}_i \delta_i \;+\; O(\delta_i^{-1}) \\
\beta U_{\rm corr}/N &= \underbrace{+c_1 R_{\widehat{0}\widehat{0}} \Lambda^2}_{\beta u_\Delta/N} - c_2 \Lambda^2 \sum_{i=1..3} (3/2)\, \mathcal{R}_i \delta_i \;+\; (1/24) \beta m c^2 \sum_{i=1..3} \mathcal{R}_i L_i^2 \;+\; O(\delta_i^{-1}) \\
C_{V\rm corr}/N &= \underbrace{-c_1 R_{\widehat{0}\widehat{0}} \Lambda^2}_{c_\Delta/N} + c_2 \Lambda^2 \sum_{i=1..3} (3/4) \mathcal{R}_i \delta_i \;+\; O(\delta_i^{-1})
\end{aligned} \tag{9}
$$

I now highlight several peculiar features of this results:

- ⋆ Perhaps the most important point to be noted is the following: the curvature term in Eq. (5), of course, has no $\hbar$ in it, it's *only* non-trivial aspect being the *perturbation of energy eigenvalues due to spacetime curvature* ($1/n_i^2$ in this case). The quantum nature of the results is nevertheless evident in the final expressions: for e.g., $\Delta \sim \hbar^2$. This is precisely the kind of non-trivial interplay alluded to in the abstract.

- ⋆ Focus on the $O({\delta_i}^0)$ term in the above expressions, which depends only on $\Delta = R_{\widehat{0}\widehat{0}} \Lambda^2$.

  This specific term is distinctive in that it does not depend on kinematical details of the system such as box dimensions L_i or mass m of the particles.

  It must be appreciated that the role of the "box" in this analysis is rather subtle: it does *not* lead just to a pure finite size effect (which can be incorporated easily [7]), but instead couples, via the boundary conditions, the Ricci tensor to the length scale $\Lambda = \hbar c/kT$, yielding the Δ term.

- ⋆ Further, s_Δ and u_Δ satisfy

$$
s_\Delta = \frac{1}{2} \beta u_\Delta \tag{10}
$$

  where

$$
s_\Delta = \frac{N\Delta}{4} = \frac{N}{4} \times \left(R_{ab} u^a u^b \right) \left(\frac{\hbar c}{kT} \right)^2 \tag{11}
$$

  Eq. (10) is a Euler relation of homogeneity two, well known from black hole thermodynamics; in particular, black hole horizons have temperature β_H^{-1}, entropy $S_{\rm bh}$ and (Komar) energy $U_{\rm bh}$ which also satisfy $S_{\rm bh} = (1/2) \beta_H U_{\rm bh}$.

Relevance of such Euler relation and area scaling of entropy for self-gravitating systems has been emphasized by Oppenheim (see 2nd reference in [3]). This relation also plays an important role in the *emergent gravity* paradigm, leading to an *equipartition law* for microscopic degrees of freedom associated with spacetime horizons [9].

- $\star$ The Δ contribution to specific heat is *negative* if the strong-energy condition ($R_{\widehat{0}\widehat{0}} \geq 0$) holds. Also, $c_\Delta = -\beta u_\Delta = -2s_\Delta$, which are again the same as the relations satisfied by a Schwarzschild black hole.

- $\star$ What is the significance of the length scale $\Lambda = \hbar c/kT$ appearing in this non-relativistic problem? The answer is not very clear at present, although it is worth mentioning that the time scale $t_o = \Lambda/c = \hbar/kT$ has recently been discussed by Haggard and Rovelli [8] as the average time a system in thermal equilibrium takes to move from a state to the next distinguishable state, thereby making t_o universal and independent of any details of the system other than its temperature.

All the above points are extremely suggestive as far as the role of Ricci correction to thermodynamic properties is concerned, and hints to the possibility that perhaps the thermodynamic properties of black holes, at least in part, might simply be an extreme manifestation of these same features in which only the Ricci term matters! However, any discussion along such lines is bound to be highly speculative, and best avoided at present. More important is the basic idea which I hope the analysis to convey: *study of thermal effects in presence of spacetime curvature can give new insights into thermodynamic aspects of gravity.* This should motivate further study of thermal systems in curved spacetime along the suggested route. Note that the important features associated with the Ricci corrections would not appear if: (i) one studies the problem using *classical* statistical mechanics – since then the modification of energy eigenvalues, $1/n^2$ in present case, is missed), or (ii) assume a priori that *finite size corrections* would necessarily depend only on area, perimeter etc – the Ricci term here in fact does not involve dimensions of the box at all! For the result to have any deeper significance, it's main *qualitative* aspects (in the high temperature limit) must, of course, survive further generalizations (relativistic gas, different statistics etc.), insofar as the *form* of the Ricci term is concerned. Some preliminary calculations do seem to point to this [10].

It is pertinent to ask: *What aspects of the above analysis could be relevant for black hole thermodynamics?* It might be that we are just beginning to understand the true significance of spacetime curvature for behaviour of thermal systems, and once we understand it better, the answer might appear to be very Holmes-*esque*: elementary . . . and perhaps inevitable.

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