EP2110: INTRODUCTION TO MATHEMATICAL PHYSICS
Jul-Nov 2019

Dawood Kothawala
Department of Physics, IIT Madras*

A. Properties of Linear Vector Spaces

A LVS $\mathbb{V}$, over a field $\mathbb{F} = \mathbb{C}$, is characterised by the properties given below. The elements $|a\rangle, |b\rangle, |c\rangle \ldots \in \mathbb{V}$, and the *scalars* $\alpha, \beta, \ldots \in \mathbb{C}$. The addition of elements is represented by $\oplus$, and the multiplication with scalars by $\odot$.

LVS1: *Commutativity of Addition*:

$$|a\rangle \oplus |b\rangle = |b\rangle \oplus |a\rangle$$

LVS2: *Associativity of Addition*:

$$\Big(|a\rangle \oplus |b\rangle\Big) \oplus |c\rangle = |a\rangle \oplus \Big(|b\rangle \oplus |c\rangle\Big)$$

LVS3: *Null element*:

There exists an element $|\Omega\rangle \in \mathbb{V}$ – called the null or zero vector – such that:

$$|a\rangle \oplus |\Omega\rangle = |\Omega\rangle \oplus |a\rangle = |a\rangle$$

LVS4: *Additive Inverse*:

For every $|a\rangle$, there exists an element $|-a\rangle \in \mathbb{V}$, such that:

$$|a\rangle \oplus |-a\rangle = |\Omega\rangle = |-a\rangle \oplus |a\rangle$$

LVS5: *Multiplicative Identity*:

$$1 \odot |a\rangle = |a\rangle$$

LVS6: *Associativity of Scalar Multiplication*:

$$(\alpha\beta) \odot |a\rangle = \alpha \odot \Big(\beta \odot |a\rangle\Big)$$

LVS7: *Distributivity of Scalar Multiplication over Vector Addition*:

$$\alpha \odot \Big(|a\rangle + |b\rangle\Big) = \alpha \odot |a\rangle + \alpha \odot |b\rangle$$

LVS8: *Distributivity of Scalar Multiplication over Scalar Addition*:

$$\Big(\alpha + \beta\Big) \odot |a\rangle = \alpha \odot |a\rangle + \beta \odot |b\rangle$$

---

*Electronic address: dawood@iitm.ac.in

EP2110: INTRODUCTION TO MATHEMATICAL PHYSICS
Jul-Nov 2019

Dawood Kothawala
Department of Physics, IIT Madras*

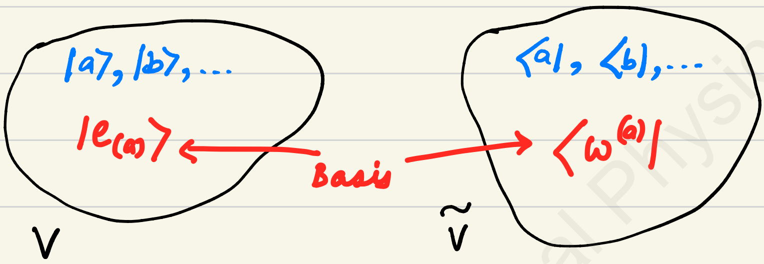
B. Dual of a LVS, Tensors, and Vector Calculus

The following handwritten notes are intended to give you a clearer understanding of the notion of a dual space $\tilde{V}$ as a space of all linear functionals acting on the elements of a given LVS V. If an inner product is defined on V, an isomorphism can be established between V and $\tilde{V}$, and the attached notes should indicate to you how. (This isomorphism, of course, depends on the choice of the inner product in V.)
You must use these notes in coordination with the discussion in the lectures, and not as a replacement for it! In case you want any clarification(s), or you encounter any error in the notes, drop me an email.

---

*Electronic address: `dawood@iitm.ac.in`

Dual Space



* $|a\rangle = a^i |e_{(i)}\rangle$
* $\langle a| = a_i \langle \omega^{(i)}|$
* $a_i = (a^k)^* g_{ki}$
* $\langle \omega^{(a)} | e_{(b)} \rangle = \delta^a_b$

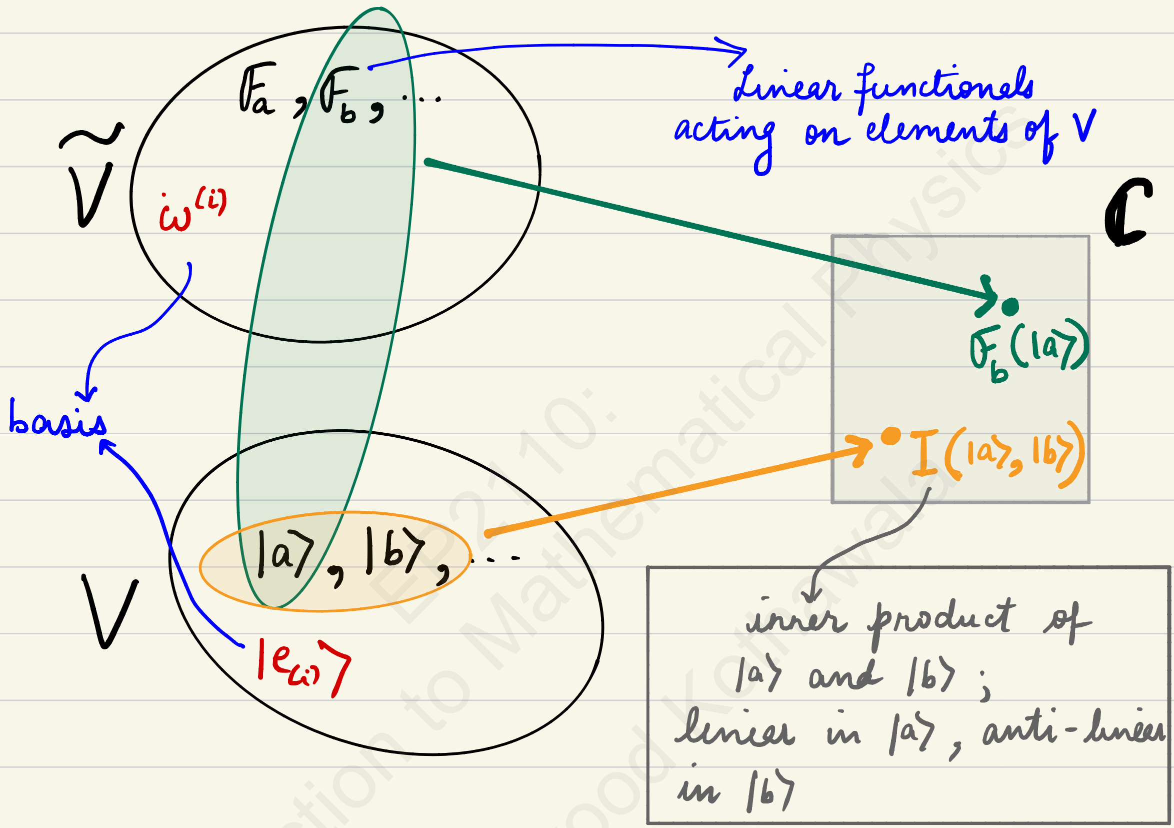
Check points :

* $\langle a| = (a^i)^* \langle e_{(i)}|$

$\Rightarrow \quad a_k \langle \omega^k| = (a^i)^* \langle e_i|$

$\Rightarrow \quad g_{ik} \langle \omega^k| = \langle e_i|$

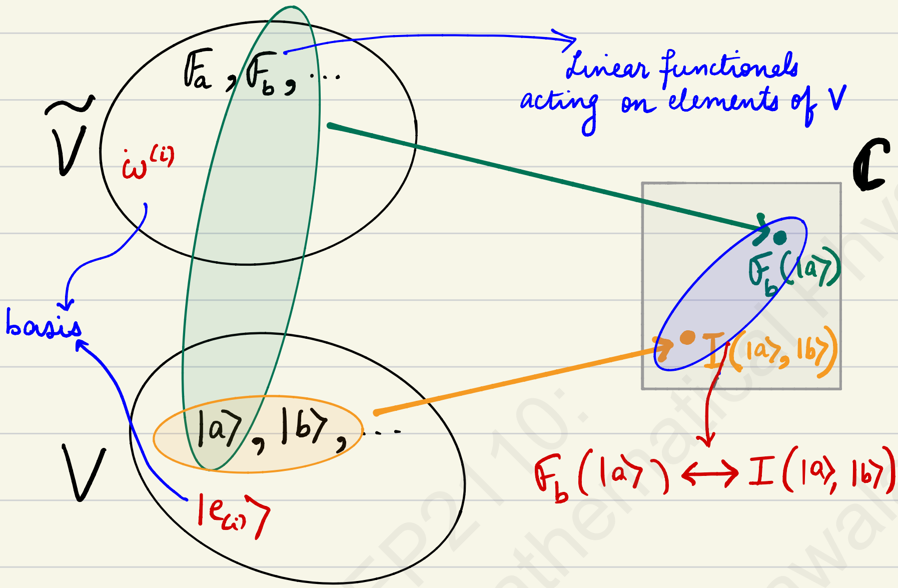
Dual Space



* An isomorphism can be established betn V and $\tilde{V}$ by demanding $\omega^{(a)}(|e_{(b)}\rangle) = \delta^a_b$

* The above isomorphism is, however, NOT NATURAL since it depends on the choice of basis [this requires some thought!]

Dual Space



linear fn — basis

$$\mathcal{A} = a_i\, \omega^{(i)}$$

$$\boxed{\mathcal{A}(|e_{(k)}\rangle) = a_k}$$

demanding $\omega^{(i)}(|e_{(k)}\rangle) = \delta^i_k$

linear in $|e_{(k)}\rangle$

$$|a\rangle = a^i |e_{(i)}\rangle$$

$$I(|a\rangle, |e_k\rangle) = a^i g_{ki}$$

$$g_{ki} \underset{\text{def.}}{=} I(|e_{(i)}\rangle, |e_{(k)}\rangle)$$

$$\boxed{I(|e_k\rangle, |a\rangle) = (a^i)^* g_{ij}}$$

$$\underset{\text{def}}{=} a_j$$

Identify : $a_k \leftrightarrow a_j$

hence $\mathcal{A}(e_{(k)}) = \langle a | e_{(k)} \rangle$

$$= I(|e_k\rangle, |a\rangle)$$

This provides an isomorphism bet. $|a\rangle$ & $\mathcal{A}$; we write $\mathcal{A} = \langle a|$ and equate

$$\boxed{\mathcal{A}(|b\rangle) = \langle a | b \rangle}$$

Tensors :

$$T \equiv t^{a_1 a_2 \cdots a_r}{}_{b_1 b_2 \cdots b_s}\; e_{(a_1)} \otimes \cdots \otimes e_{(a_r)} \otimes \omega^{(b_1)} \otimes \cdots \otimes \omega^{(b_s)}$$

components (in a) Basis

The abstract tensor

Appearance in Basic physics

* Gradient $\nabla \leftrightarrow \partial_i$

* $\delta^i{}_j$ → Kronecker delta * ε_{abc} ; ε^{abc} → Levi-Civita tensor

* $V = \frac{1}{r} q + \frac{1}{r^2} p_i n^i + \frac{1}{r^3} Q_{ij} n^i n^j + \frac{1}{r^4} D_{ijk} n^i n^j n^k + \cdots$

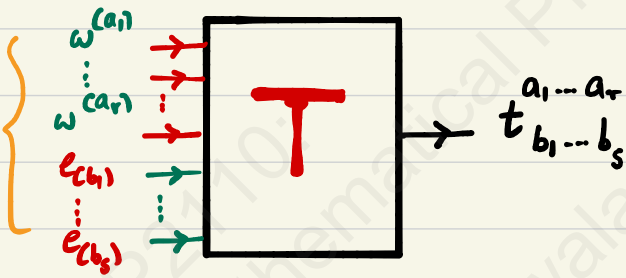
  Multipole expansion

* Maxwell Stress tensor

$$\mathbf{S} \leftrightarrow S_{ij}$$

* Visualizing Tensors as "Machines"

$$T \equiv t^{a_1 a_2 \cdots a_r}_{b_1 b_2 \cdots b_s}\, e_{(a_1)} \otimes \cdots \otimes e_{(a_r)} \otimes \omega^{(b_1)} \otimes \cdots \otimes \omega^{(b_s)}$$



$$T\left[\omega^{(a_1)}, \cdots \omega^{(a_r)};\, e_{(b_1)}, \cdots e_{(b_s)}\right]$$

INPUT SLOTS OF THE MACHINE T

$$= t^{\bar{a}_1 \bar{a}_2 \cdots \bar{a}_r}_{\bar{b}_1 \bar{b}_2 \cdots \bar{b}_s}\, e_{(\bar{a}_1)}\left(\omega^{(a_1)}\right) \cdots e_{(\bar{a}_r)}\left(\omega^{(a_r)}\right) \times \omega^{(\bar{b}_1)}\left(e_{(b_1)}\right) \cdots \omega^{(\bar{b}_s)}\left(e_{(b_s)}\right)$$

$\delta^{a_1}_{\bar{a}_1}$ etc.

$\delta^{\bar{b}_1}_{b_1}$

$$= t^{a_1 a_2 \cdots a_r}_{b_1 b_2 \cdots b_s} \longrightarrow \text{OUTPUT}$$

EXAMPLES

$$|e_{(1)}\rangle \equiv \begin{pmatrix}1\\0\\0\end{pmatrix} ; \quad |e_{(2)}\rangle \equiv \begin{pmatrix}1\\1\\0\end{pmatrix} ; \quad |e_{(3)}\rangle \equiv \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\langle \omega^{(1)}| \equiv (a_1 \;\; a_2 \;\; a_3) \quad ; \quad \langle \omega^{(2)}| \equiv (b_1, b_2, b_3)$$

etc.

$$\left.\begin{aligned} a_1 &= 1 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0 \end{aligned}\right\} \begin{aligned} a_2 &= -1 \\ a_3 &= 0 \end{aligned} \Rightarrow \boxed{\langle \omega^1| = (1, -1, 0)}$$

$$\left.\begin{aligned} b_1 &= 0 \\ b_1 + b_2 &= 1 \\ b_1 + b_2 + b_3 &= 0 \end{aligned}\right\} \begin{aligned} b_2 &= 1 \\ b_3 &= -1 \end{aligned} \Rightarrow \boxed{\langle \omega^2| = (0, 1, -1)}$$

$$\left.\begin{aligned} c_1 &= 0 \\ c_1 + c_2 &= 0 \\ c_1 + c_2 + c_3 &= 1 \end{aligned}\right\} \begin{aligned} c_2 &= 0 \\ c_3 &= 1 \end{aligned} \Rightarrow \boxed{\langle \omega^3| = (0, 0, 1)}$$

NOTE: These are different from $\langle e_1|, \langle e_2|, \langle e_3|$

* VECTOR CALCULUS :

Let $g_{ab} = C_{ab}$ = const. matrix

* $q^i = \varepsilon^{iab} \partial_a v_b$ → curl in Cartesian coords.

* $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) \equiv \varepsilon_{ijk}\, a^j\, \varepsilon^{klm}\, b_l\, c_m$

$$= (a^m c_m)\, b_i - (a^m b_m)\, c_i$$

$$\Rightarrow \boxed{\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\, \underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\, \underset{\sim}{c}}$$

Similar identities can be proved in curvilinear coords., provided the Levi-Civita tensor is aptly defined.

CURL - example

$$v^a \xrightarrow{g_{ab}=\delta_{ab}} v_a \longrightarrow \varepsilon^{ijk}\partial_j v_k = q^i$$

$$q^1 = \varepsilon^{123}\left(\partial_2 v_3 - \partial_3 v_2\right) = \partial_2 v_3 - \partial_3 v_2$$

* $$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \varepsilon^{ijk}\partial_j q_k$$

defn of q^m

$$= \varepsilon^{ijk}\partial_j\left[\delta_{km}\,\varepsilon^{mab}\partial_a v_b\right]$$

$$\delta_{km}\,q^m = q_k$$

$$= \varepsilon^{ijk}\varepsilon^{kab}\,\partial_j\partial_a v_b$$

$$= \left(\delta^{ia}\delta^{jb} - \delta^{ib}\delta^{ja}\right)\partial_j\partial_a v_b$$

$$[\nabla\times(\nabla\times v)]^i = \partial_b\partial_i v_b - \partial_a\partial_a v_i$$

$$\Rightarrow \boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \nabla^2\vec{v}}$$

* Geometry

* coordinate basis :

$$d\vec{r} = dx^i \, \vec{e}_{(i)} \Rightarrow \boxed{\vec{e}_{(i)} = \frac{\partial \vec{r}}{\partial x^i}}$$

* $$g_{ab} = \vec{e}_{(a)} \cdot \vec{e}_{(b)}$$

* $$dV = dx^1 dx^2 dx^3 \boxed{\underbrace{\left| \vec{e}_3 \cdot \left(\vec{e}_1 \times \vec{e}_2 \right) \right|}_{= \sqrt{g}}}$$

Sketch of proof:

* Defn. of determinant :

$$\det A = \varepsilon_{abc} A^a_1 A^b_2 A^c_3$$

* $$\vec{e}_3 \cdot (\vec{e}_1 \times \vec{e}_2) = \varepsilon_{abc} \, e^a_{(3)} \, e^b_{(1)} \, e^c_{(2)} = \det(e)$$

* $$g_{ab} = \delta_{ij} \, e^i_{(a)} \, e^j_{(b)} \Rightarrow \det g = (\det e)^2$$

VECTOR CALCULUS

* Gradient "vector":

$$f_i = \frac{\partial f}{\partial x^i} \quad ; \quad f^i = g^{ij} \frac{\partial f}{\partial x^j}$$

eg: Newton's law:

$$m a^i = g^{ij} \frac{\partial V}{\partial x^j}$$

compare with components of $\vec{\nabla} V$ in Griffiths