LOOP QUANTUM COSMOLOGY:

RECENT PROGRESS

- Basic structure
  First applications

- Quantum structure of classical singularities
  Phenomenology
  Conceptual issues

Loop Quantum Gravity:

background independent, non-perturbative quantization

well-equipped for extreme physical situations:

big bang, black holes

techniques often unfamiliar, but essential for

well-defined framework

can be illustrated, tested in simple situations by

introducing symmetries: loop quantum cosmology

characteristic basic features survive reduction

far-reaching consequences

explicit models

contact to observations

gravity in Ashtekar variables: conjugate pair on space Σ

densitized triad E^a_i, $SU(2)$-connection $A^i_a = \Gamma^i_a - \gamma K^i_a$

$\Gamma(E)$: spin connection, K: extrinsic curvature, $\gamma \in \mathbb{R}$: Barbero-Immirzi parameter

gauge theory with compact gauge group (triad rotations)

plus additional constraints (diffeomorphism, Hamiltonian)

basic objects for background independent quantization:

holonomies $\quad h_e(A) = P\exp \int_e A^i_a \tau_i \dot{e}^a dt \in SU(2)$

fluxes $\quad F_S(E) = \int_S E^a_i n_a \tau^i d^2y \in su(2)$

connection representation: • spin network states associated with labeled graphs in Σ, by contracting holonomies

- Ashtekar-Lewandowski inner product
- fluxes: derivative operators, count intersection number

basic properties: • huge, non-separable Hilbert space

- holonomies well-defined, not A itself
- fluxes with discrete spectrum, discrete geometry

constraints (Einstein's equations) complicated

simplification symmetries

isotropy: metric $ds^2 = -dt^2 + a(t)^2 \, d\Omega_k^2$

densitized triad $E^a_i = p \, \Lambda^I_i X^a_I$

connection $A^i_a = c \, \Lambda^i_I \omega^I_a$

ω^I, X_I: left-invariant 1-forms, vector fields (from symmetry group)

Λ: su(2)-triad (pure gauge)

$|p| = a^2$ (sgn p: orientation), $c = \frac{1}{2}(k - \gamma \dot{a})$

Hamiltonian constraint

$$12 \gamma^{-2} \left[c(c-k) + (1+\gamma^2) k^2/4 \right] \sqrt{|p|} = 8\pi G \, H_{matter}(p, \phi, \ldots)$$

imposes Friedmann equation $3(\dot{a}^2 + k^2) a = 8\pi G \, a^3 \rho_{matter}(a)$

Wheeler-DeWitt quantization:

Schrödinger representation, wave functions $\psi(a)$

2nd order differential equation

loop quantum cosmology: direct link to full theory

symmetric states by definition supported on invariant connections

holonomies $h_I = \exp(c\Lambda_I^i \tau_i \int \omega^I) = \cos\frac{\mu c}{2} + 2\Lambda_I^i \tau_i \sin\frac{\mu c}{2}$

$\mu \in \mathbb{R}$: parameter length of edge

SU(2)-gauge invariant isotropic states:

$$\psi(c) = \sum_{\mu \in I} \psi_\mu e^{i\mu c/2} \qquad I \subset \mathbb{R} \text{ countable}$$

inner product $\langle \mu | \mu' \rangle = \delta_{\mu\mu'}$ $\qquad \langle c | \mu \rangle = e^{i\mu c/2}$

basic operators $\widehat{e^{i\mu' c/2}} |\mu\rangle = |\mu + \mu'\rangle$

$$\hat{p} |\mu\rangle = \frac{1}{6}\gamma \ell_P^2 \mu |\mu\rangle$$

properties:
- non-separable Hilbert space
- only holonomy $e^{i\mu c/2}$ well-defined, no c-operator
- discrete flux spectrum: normalizable eigenstates

as in full theory

A. Ashtekar, MB, J. Lewandowski: Adv. Theor. Math. Phys. 7 (2003) 233

very different from Wheeler-DeWitt: inequivalent representation further consequences: (i) discrete evolution

quantized Friedmann equation becomes difference equation

reason: no c-operator, use holonomies

difference to Wheeler-DeWitt important at small scales

cosmological singularities removed in homogeneous models

Example: isotropy

$$\left(V_{\mu+5}-V_{\mu+3}\right)e^{ik}\psi_{\mu+4}(\phi)-\left(2+\gamma^2k^2\right)\left(V_{\mu+1}-V_{\mu-1}\right)\psi_\mu(\phi)+e^{-ik}\left(V_{\mu-3}-V_{\mu-5}\right)\psi_{\mu-4}(\phi)$$
$$=-\tfrac{4}{3}G\gamma^3\ell_P^2\,\hat{H}_{\text{matter}}(\mu)\,\psi_\mu(\phi)$$

$\psi_\mu(\phi)$: wave function, $V_\mu=\left(\frac{1}{6}\gamma\ell_P^2|\mu|\right)^{3/2}$: volume eigenvalues

$\hat{H}_{\text{matter}}(\mu)$: matter Hamiltonian for ϕ

recurrence relation, does not break down at

classical singularity $\mu=0$, evolves to negative μ

essential: $\hat{H}_{\text{matter}}(0)=0$ e.g. $\hat{H}_{\text{matter}}=\frac{1}{2}\widehat{a^{-3}}\otimes\hat{p}_\phi^2+\hat{V}\otimes W(\phi)$

(iii) finite inverse scale factor operator

reason: $\hat{p}$ has discrete spectrum containing zero, no inverse

instead: rewrite

$$a^{-3} = \left(\frac{1}{2\pi\gamma G}\{c, |p|^{3/4}\}\right)^6 = \left[\frac{1}{3\pi\gamma G}\sum_I \mathrm{tr}\left(\Lambda_I^i \tau_i h_I \{h_I^{-1}, \sqrt{V}\}\right)\right]^6$$

$h_I = e^{c\Lambda_I^i \tau_i}$

quantize using $\hat{p}$, holonomy operators,

turn Poisson bracket to commutator

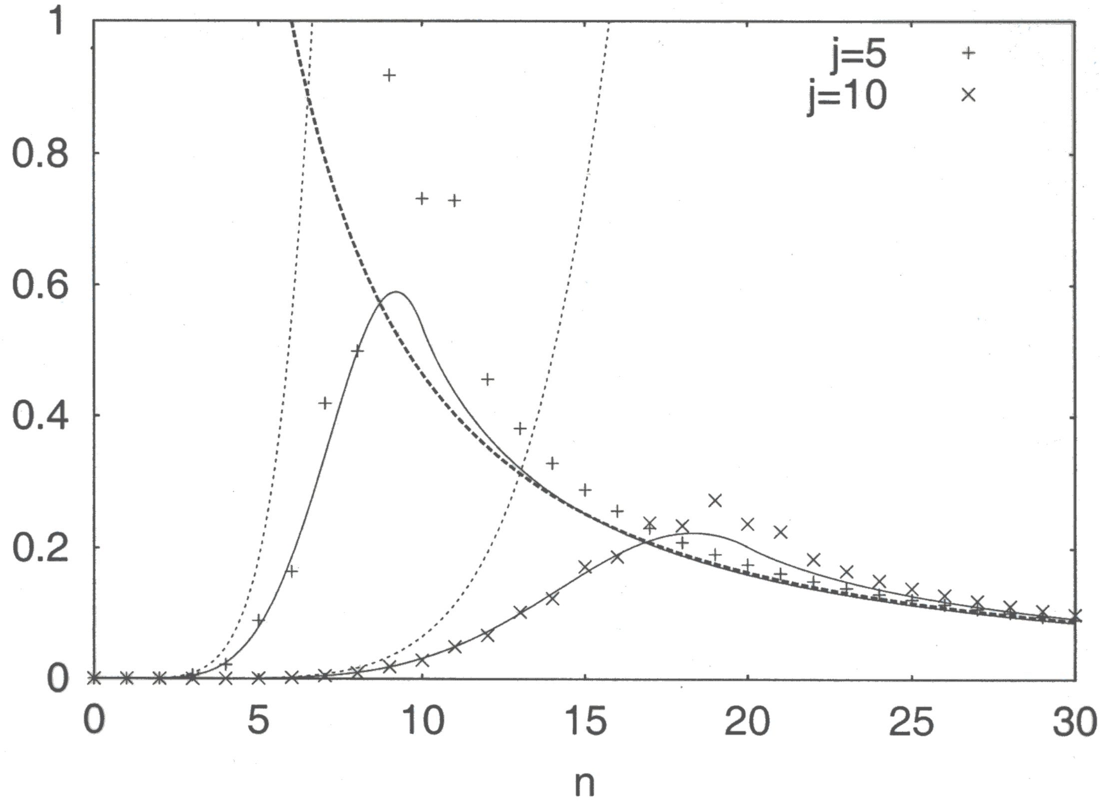
no inverse V needed; well-defined, finite quantization

non-basic object: quantization ambiguities

e.g.

$$a^{-3} = \left[\frac{1}{2\pi\gamma G j(j+1)(2j+1)}\sum_I \mathrm{tr}_j \left(\Lambda_I^i \tau_i h_I \{h_I^{-1}, \sqrt{V}\}\right)\right]^6$$

$j \in \frac{1}{2}\mathbb{N}$ determines position of peak: $\mu_{max} \approx 2j$



$$(a^{-3})_{j,n} = \left[12(j(j+1)(2j+1)\gamma l_{\mathrm{P}}^2)^{-1}\sum_{k=-j}^{j} k\sqrt{V_{\frac{1}{2}(|n+2k|-1)}}\right]^6$$

$$\simeq V^{-1}_{\frac{1}{2}(|n|-1)}\, p(|n|/2j)^6 \qquad \text{(solid lines)}$$

with

$$p(q) = \frac{8}{77}q^{\frac{1}{4}}\left(7\left((q+1)^{\frac{11}{4}} - |q-1|^{\frac{11}{4}}\right) - 11q\left((q+1)^{\frac{7}{4}} - \operatorname{sgn}(q-1)|q-1|^{\frac{7}{4}}\right)\right)$$

effective density:

$$(a^{-3})_{j,n(a)} = a^{-3}p(3a^2/\gamma l_{\mathrm{P}}^2 j)^6 \simeq \frac{12^6}{7^6}(\gamma l_{\mathrm{P}}^2 j/3)^{-\frac{15}{2}}a^{12} \quad \text{for} \quad a^2 \ll \frac{1}{3}\gamma l_{\mathrm{P}}^2 j$$

eigenvalues: $\left(\widehat{a^{-3}}\right)^{(j)}_{\mu} = \left[\frac{1}{12\gamma\ell_P^2\, j(j+1)(2j+1)} \sum_{k=-j}^{j} k \sqrt{V_{|\mu+2k|}}\right]^6$

small-volume properties:

$\mu=0$: $\left(\widehat{a^{-3}}\right)^{(j)}_{\mu=0} = 0$ (sum of odd $k\sqrt{V_{2|k|}}$)

implies vanishing matter Hamiltonians on $|\mu=0\rangle$

e.g. $\hat{H}_\phi(0) = \frac{1}{2}\left(\widehat{a^{-3}}\right)^{(j)}_{\mu=0} \hat{p}_\phi^2 + V_{\mu=0} W(\phi) = 0$

$0 \le \mu \lesssim 2j$: $(a^{-3})^{(j)}_{\mathrm{eff}}(a) := \left(\widehat{a^{-3}}\right)^{(j)}_{\mu(a)=a^2/6\gamma\ell_P^2}$

increases with a

modified Friedmann equation via matter Hamiltonian

inflation

$\mu \approx 2j$: peak, transition to classical a^{-3}

intuitive explanation:

- peak: finite bound $a^{-3}_{\max} \sim \frac{1}{(\gamma j)^{3/2}\ell_P^3} \xrightarrow{\ell_P \to 0} \infty$

  analogous to hydrogen atom: $E_0 = -\frac{1}{2}\frac{me^4}{\hbar^2} \xrightarrow{\hbar \to 0} -\infty$

- around peak: interpolation between inflation, ordinary matter

black body radiation

spectral energy density $\rho_T(\lambda) \sim \frac{8\pi}{c}\frac{kT}{\lambda^3}$, $\lambda \gg kT$ Rayleigh-Jeans

diverges for small λ, unphysical

Planck: $\rho_T(\lambda) = 8\pi \frac{h\lambda^{-3}}{e^{\frac{hc}{kT\lambda}} - 1} = \frac{h}{\lambda^3} f\left(\frac{\lambda}{\lambda_{max}}\right)$

$\lambda_{max} = \frac{hc}{x\,kT}$, $e^x(3-x) = 3$, $f(y) = 8\pi\left[\left(\frac{3}{3-x}\right)^{1/y} - 1\right]^{-1}$

peak position by T^{-1}

effective density

classically: a^{-3} diverges

$(a^{-3})^{(j)}_{eff}(a) = a^{-3}\,p(a^2/a^2_{max})^6$ $\qquad a_{max} = \sqrt{\gamma j/3}\,\ell_p$

$p(q) = \frac{8}{77} q^{1/4}\left[7\left((q+1)^{11/4} - |q-1|^{11/4}\right) - 11q\left((q+1)^{7/4} - \mathrm{sgn}(q-1)|q-1|^{7/4}\right)\right]$

subject to ambiguities

peak position by j

de Sitter: assume $p(q)$ such that $(a^{-3})_{eff} = \Lambda_{eff}\,a^3 \rightarrow \Lambda_{eff} \propto a_{max}^{-6}$

?

$T_{dS} = \frac{\sqrt{\Lambda}}{2\sqrt{3}\pi} \propto a_{max}^{-3} \propto j^{-3/2}\,\ell_p^{-3}$

$V_{max}/V \propto \frac{1}{VT}$, $(a^{-3})^{(j)}_{eff}(V) = V^{-1} g(VT)$

recent applications

(i) quantum structure of classical singularities

dynamics by difference equation, does not break down

new branch (beyond singularity) with reversed orientation

general scheme in homogeneous models

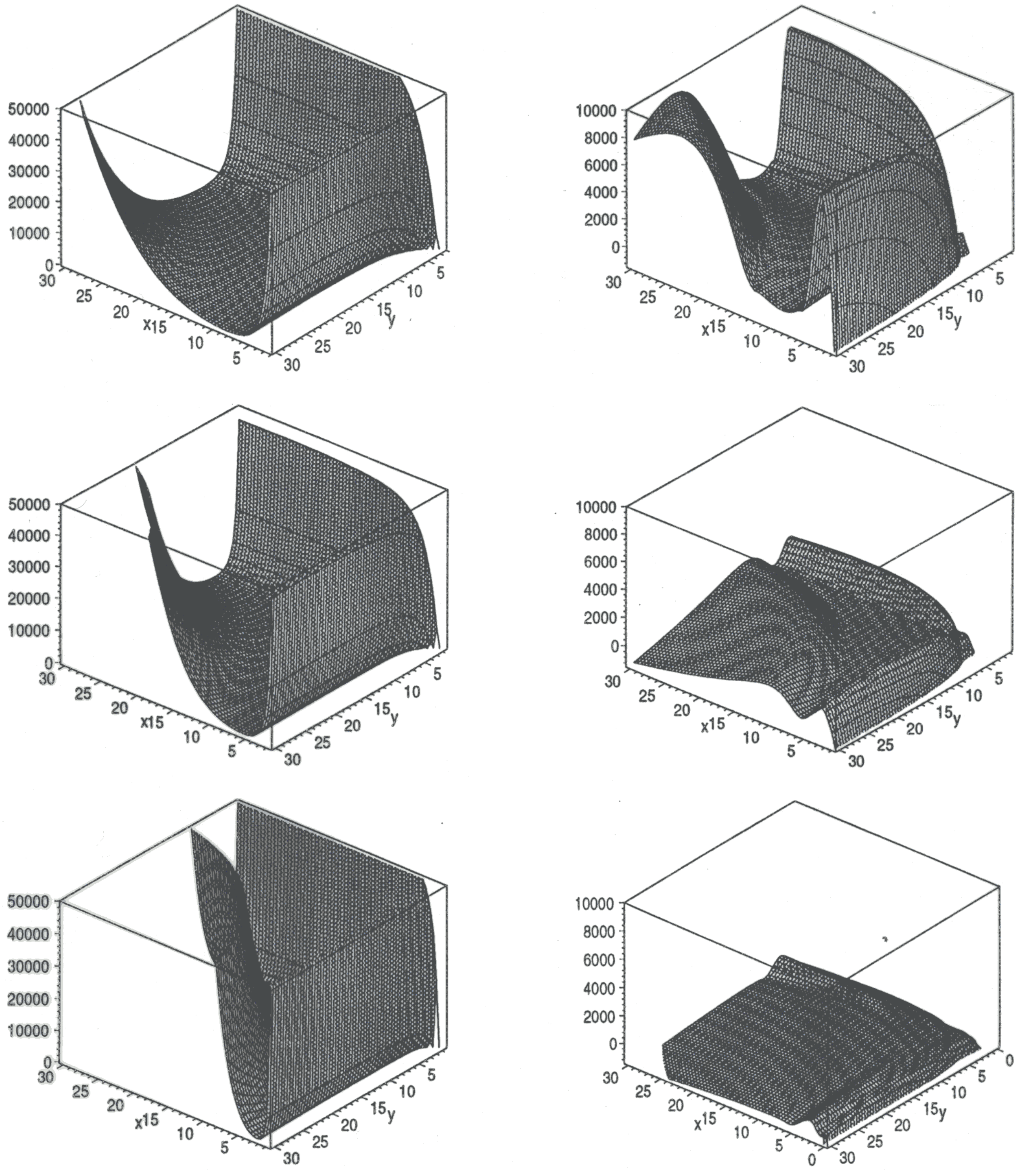
toward general singularity: BKL scenario

- approximate space as composed of almost homogeneous patches
- each evolves following most general homogeneous behavior: Bianchi IX
- chaos: initially similar patches rapidly depart from each other,

  need to be subdivided to maintain approximation

fragmentation without bound

inconsistent with discrete geometry



Classical (left) and non-perturbative quantum (right, $j = 3$) potential for the Bianchi IX universe at constant volume $V_1 = (\frac{13}{2}\gamma\ell_{\mathrm{P}}^2)^{\frac{3}{2}}$ (top), $V_2 = (5\gamma\ell_{\mathrm{P}}^2)^{\frac{3}{2}}$ (middle), and $V_3 = (3\gamma\ell_{\mathrm{P}}^2)^{\frac{3}{2}}$ (bottom).

MB, Date: gr-qc/0311003
Phys. Rev. Lett.

but: chaos from unbounded curvature, purely classical

loop effects: Bianchi IX potential bounded

walls disappear with shrinking volume

chaos stops at discrete scale at the latest

MB, G. Date: gr-qc/0311003
Phys. Rev. Lett.

classical picture of bounces: evolution from

matter Hamiltonian $\frac{1}{2}(a^{-3})_{\mathrm{eff}}\, p_\phi^2 + a^3 W(\phi)$

$$3\,\frac{\dot a^2 + k}{a^2} = 8\pi G\left[\tfrac{1}{2}(a^{-3})_{\mathrm{eff}}\,\dot\phi^2 + W(\phi)\right]$$

$$\ddot\phi = \underbrace{\dot a\,\dot\phi\,\frac{d}{da}\log(a^{-3})_{\mathrm{eff}}}_{\text{friction / antifriction depending on sgn } \dot a \frac{d}{da}\log(a^{-3})_{\mathrm{eff}}} - a^3 (a^{-3})_{\mathrm{eff}}\, W'(\phi)$$

evolution toward bounce: $\dot a < 0$

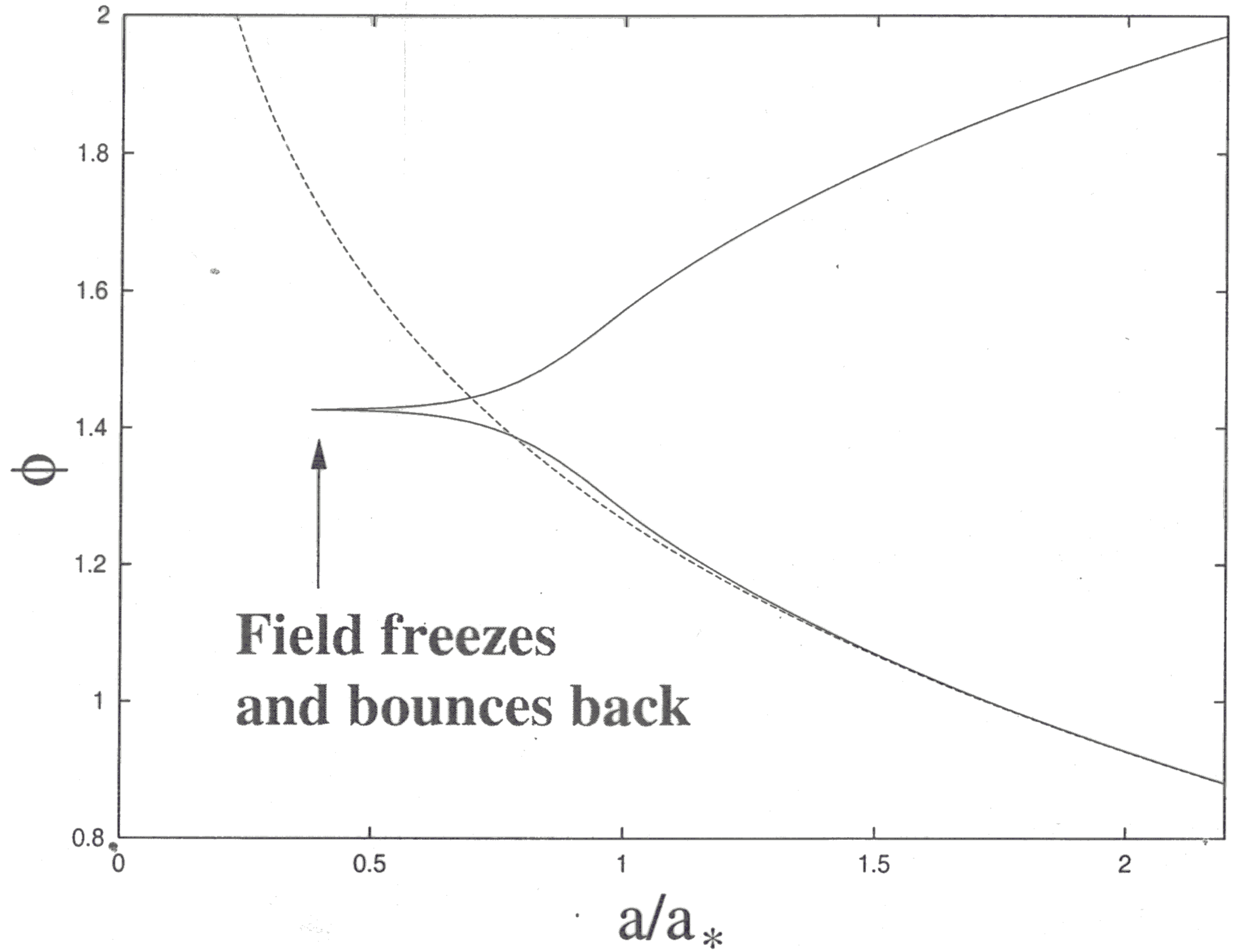
classical: $\frac{d}{da}\log a^{-3} = -\frac{3}{a} < 0$ antifriction, ϕ diverges

effective: $\frac{d}{da}\log(a^{-3})_{\mathrm{eff}} > 0$ below peak

friction, ϕ freezes, $\dot\phi \sim 0$, de Sitter bounce

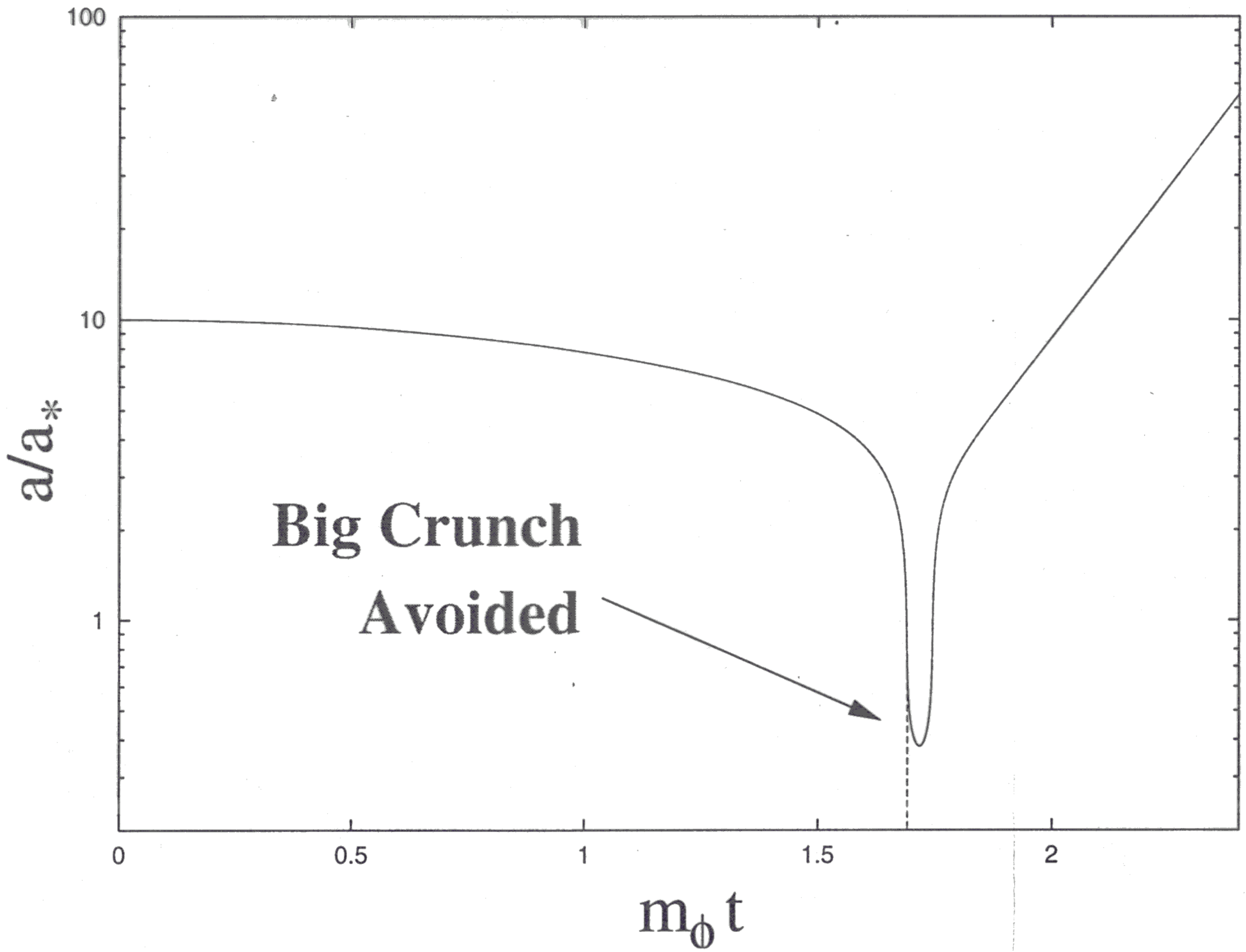
P. Singh, A. Toporensky: gr-qc/0312110

Behavior of ϕ in LQC



$j = 100, \quad m_\varphi = 0.1\ M_{pl}, \quad a_i = 10\ a_*$

Big crunch avoidance in LQC



$j = 100, \quad m_\phi = 0.1\ M_{pl}, \quad a_i = 10\ a_*$

after bounce: $\dot{a} > 0$ classically friction (e.g. slow-roll)

effective: anti-friction, ϕ driven up its potential

(ii) phenomenology

increasing $(a^{-3})_{\mathrm{eff}}(a)$ implies accelerated expansion

stops automatically after peak

but: perturbation theory unstable, structure formation unclear

second effect: ϕ increases early reaching large values

then turns around, drives slow-roll inflation

around turning point $\dot{\phi} \approx 0$: slow-roll violated

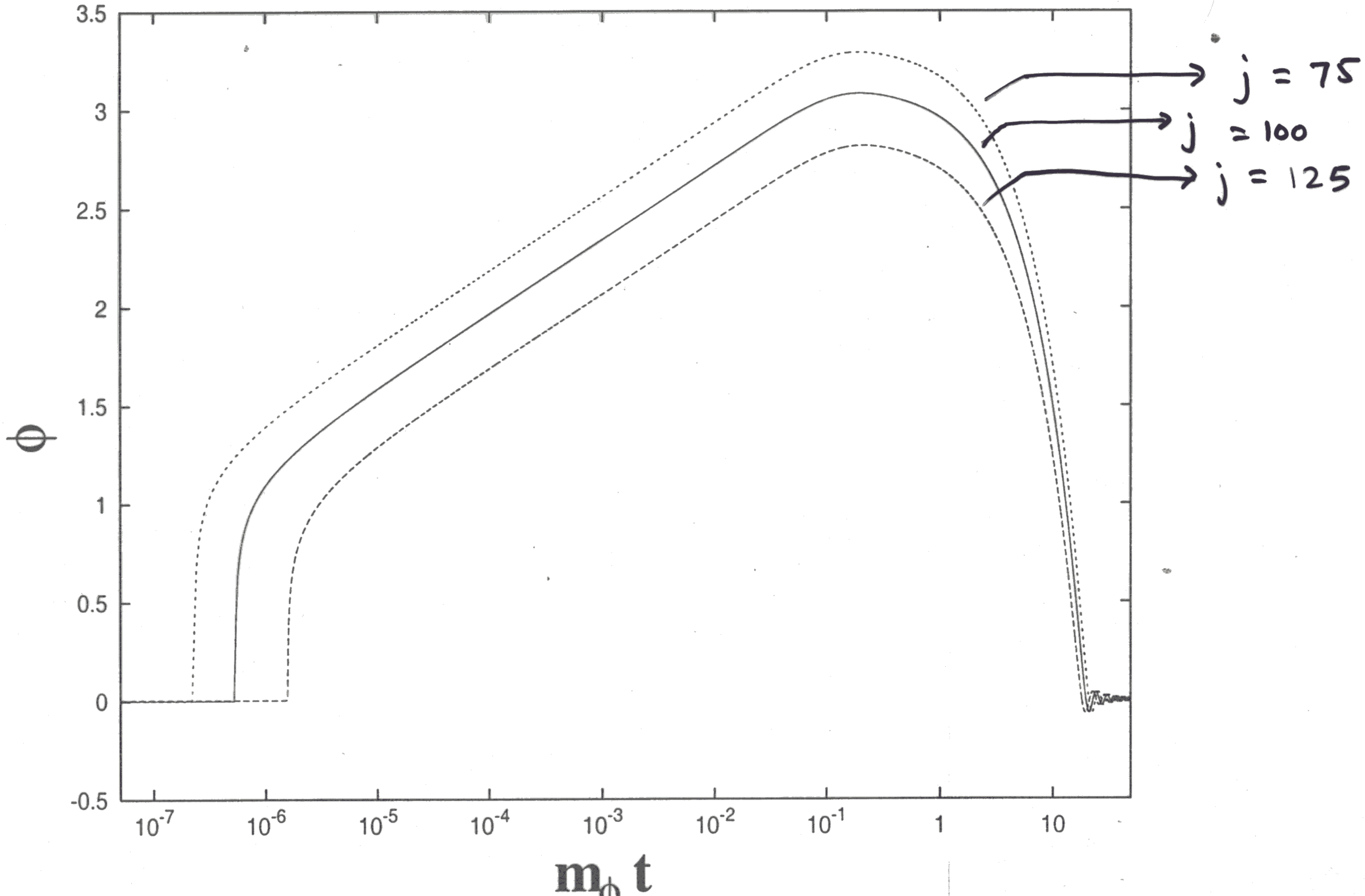
observable consequences if at right scales:

suppression of power

running of spectral index

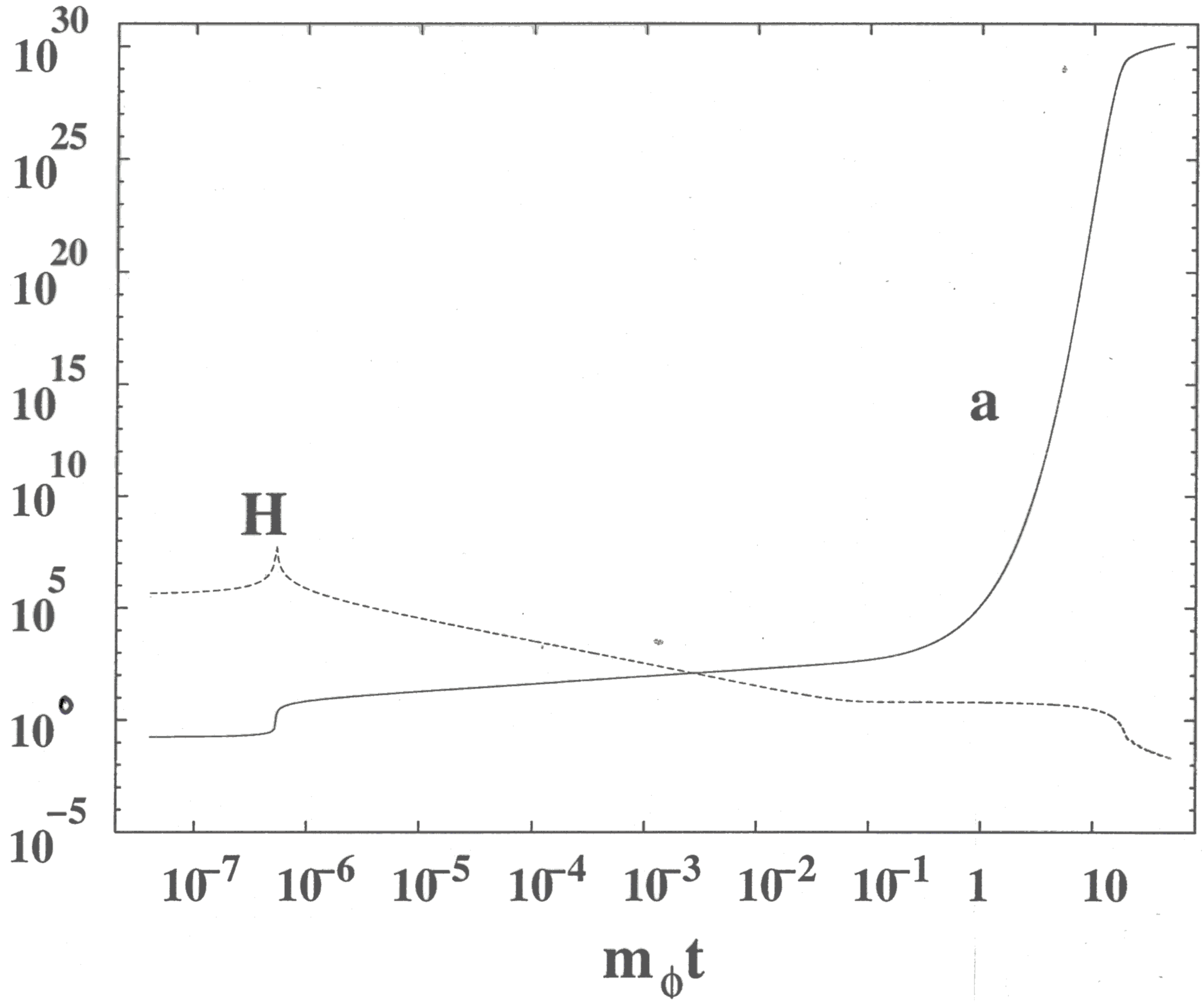
S. Tsujikawa, P. Singh, R. Maartens: astro-ph/0311015

Quantum Kick of the Inflaton

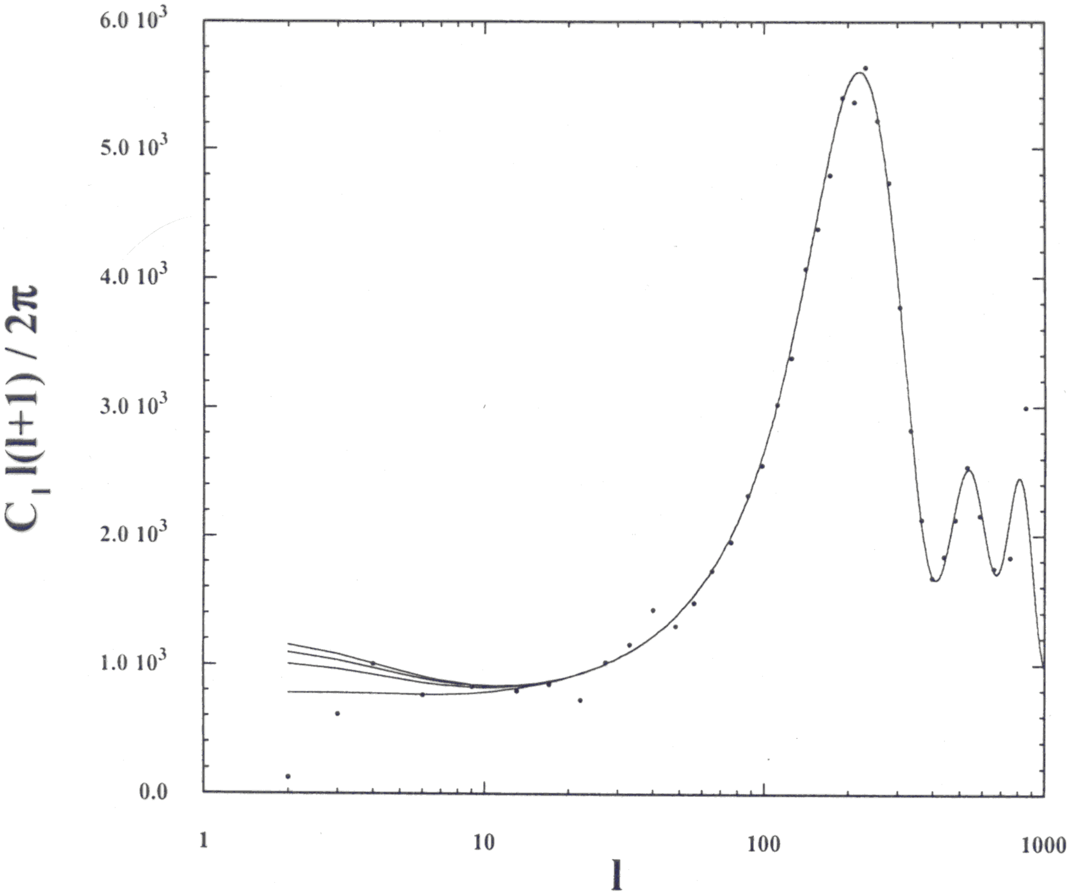


$$\frac{\dot{\phi}_i}{m_\phi M_{pl}} = 2 \ , \quad \frac{m_\phi}{M_{pl}} = 10^{-6}$$

H and a in LQC



$j = 100$, $a_i = \sqrt{\gamma}\, l_{Pl}$, $\frac{\dot{\varphi}_i}{m_\varphi M_{Pl}} = 2$, $\frac{m_\varphi}{M_{Pl}} = 10^{-6}$



The CMB angular power spectrum with loop quantum inflation effects. From top to bottom, the curves correspond to (i) no loop quantum era (standard slow-roll chaotic inflation), (ii) $\bar{\alpha} = -0.04$ for $k \le k_0 = 2 \times 10^{-3}\mathrm{Mpc}^{-1}$, (iii) $\bar{\alpha} = -0.1$ for $k \le k_0$, and (iv) $\bar{\alpha} = -0.3$ for $k \le k_0$. There is some suppression of power on large scales due to the running of the spectral index.

S. Tsujikawa, P. Singh, R. Maartens

astro-ph/0311015

(iii) perturbative corrections

expectation value of Hamiltonian constraint operator in coherent state $|\psi\rangle = \sum_{m\in\mathbb{Z}} e^{-\frac{1}{2}\frac{(m-N)^2}{d^2}} e^{-\frac{i}{2}c_0(m-N)} |m\rangle$

peaked at $(c,p) = (c_0, \frac{1}{6}\gamma\ell_P^2 N)$ with spread $1 \ll d \ll N$

$$\langle\hat{H}\rangle = H_{\mathrm{class}} + O(d^{-2}\sqrt{p_0}) + O(c_0^4\sqrt{p_0})$$

(asymptotic series)

with $H_{\mathrm{class}} = -12\gamma^{-2} c_0^2 \sqrt{p_0}$, $\quad p_0 = \frac{1}{6}\gamma\ell_P^2 N$

two types of corrections:

quantum uncertainty, higher curvature

corresponds to isotropic terms of effective action

more freedom: no unique coherent state / vacuum

(squeezing parameter d)

can be restricted by studying further quantities

A. Ashtekar, MB, J. Willis

(iv) conceptual issues

physical inner product, quantum observables

several strategies: Dirac Analysis, spin foam ideas

G. Hossain: Class. Quantum Grav. 21 (2004) 179

MB, K. Noui, A. Perez, K. Vandersloot

not yet well-developed

important to understand solutions:

infinitely many even in isotropic vacuum case

all but two rapidly oscillating (below Planck scale)

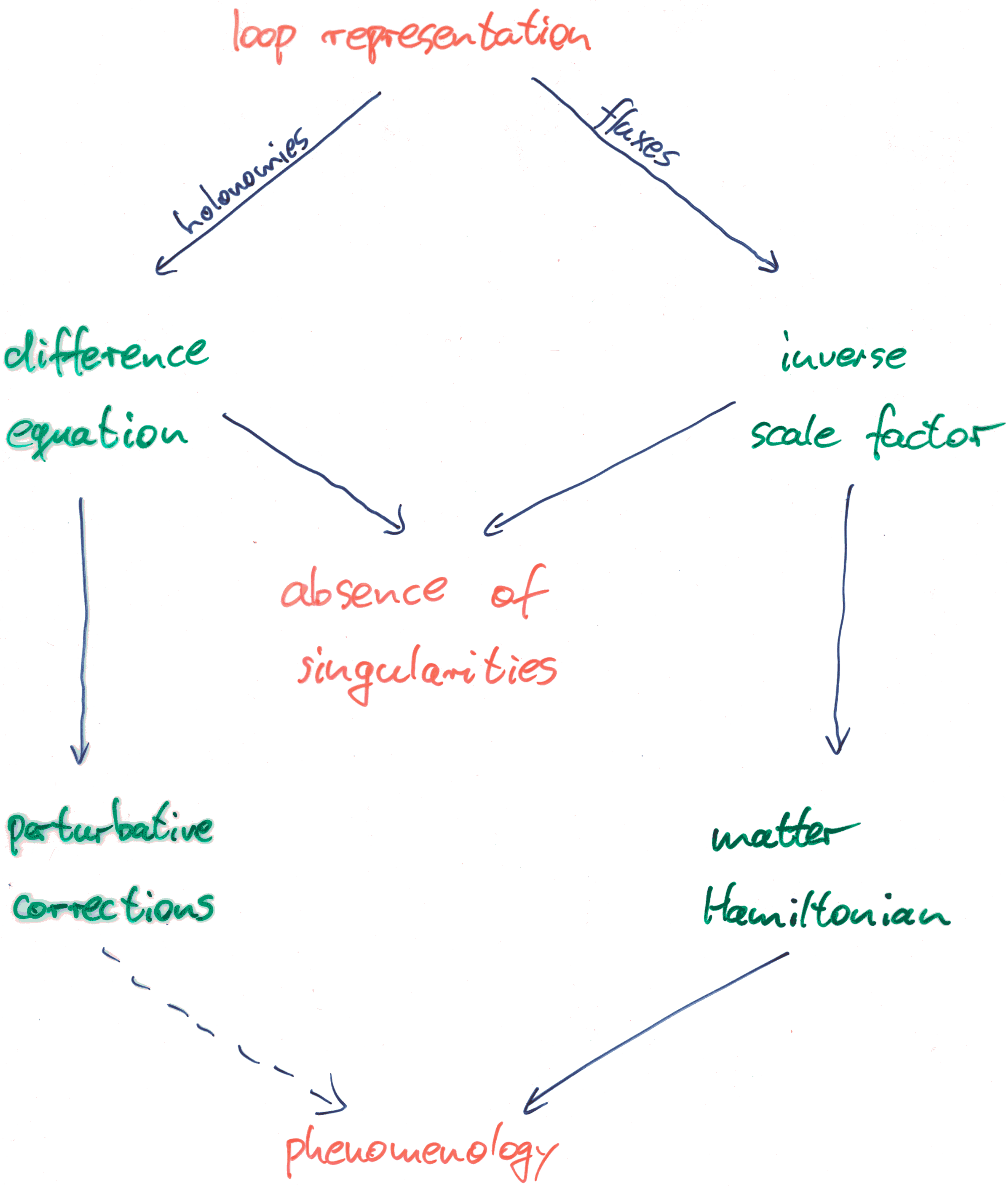
no continuum approximation

remaining two solutions correspond to

Wheeler-DeWitt solutions at large volume

further restricted (dynamical initial condition)

meaning of additional solutions? superselection?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