TRAVERSABLE WORMHOLES WITH ARBITRARILY SMALL VIOLATIONS OF THE ENERGY CONDITIONS

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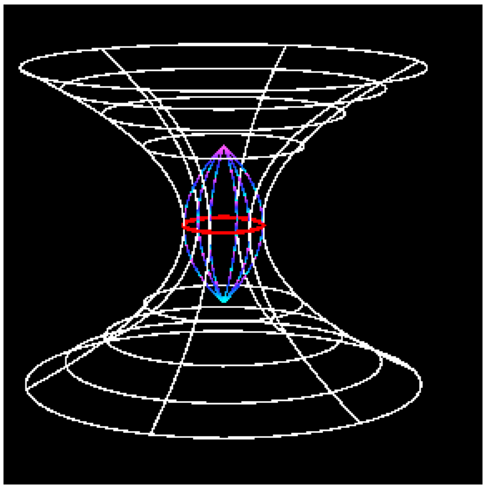
Reference :

MV, SK, ND, Phys. Rev. Lett. 90, 201102 (2003)

What is a wormhole?

• **t=constant $\theta = \frac{\pi}{2}$ section of 4D metric**

$\rightarrow$ two asymptotically flat regions connected by a bridge



• **required matter is 'exotic'**

$\rightarrow$ violates the local Energy Conditions, eg. Weak Energy Condition : $\rho \geq 0, \rho + p_i \geq 0$

$\rightarrow$ also violates the Averaged Energy Conditions , eg. Averaged Null Energy Condition : $\int_{\lambda_1}^{\lambda_2} T_{ij}\xi^i\xi^j d\lambda \geq 0$, ξ^i tangent to null geodesic

How exotic is 'exotic matter'?

• Casimir Effect [1948 (theory), 1999 (expt)]

$\rightarrow$ The force of attraction between parallel metal plates in vacuum $\rightarrow$ indirect proof of negative energy density.

• Epstein–Glaser–Yaffe theorem (1965)

$\rightarrow$ negative energy density in terms of expectation value of T_{00} over certain states is possible.

• Exotic stuff today $\rightarrow$ reality tomorrow!

$\rightarrow$ Negative group velocity (Brillouin, Sommerfeld, Garrett and McCumber(1972)), realised in 2000)

$\rightarrow$ Negative refractive index (Veselago (1963), realised in 2001)

So why not wormholes with exotic matter?

MOTIVATION AND OUTLINE

• How much exotic matter? Is there a way to quantify the amount of exotic matter?

• Do we need large amounts of exotic matter for wormholes?

• Global theorems of GR: (positive mass,area increase,singularity, topological censorship)

$\rightarrow$ Small violations of EC can evade the conclusions of these theorems?

• Quantum effects produce exotic matter :

$\rightarrow$ 'size' and 'distribution' bound by the Ford–Roman quantum inequalities

We can geometrically engineer wormholes with arbitrarily small amounts of exotic matter

Spherically symmetric, static wormholes

Line element :

$$\boxed{\mathbf{ds^2 = -e^{2\psi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega_2^2}}$$

Wormhole constraints on $b(r)$ and $\psi(r)$:

- $\frac{b(r)}{r} \leq 1$; $b(r = b_0) = b_0$ **(b_0 the throat)**
- $\frac{b(r)}{r} \to 0$ **as** $r \to \infty$ **(asymptotic flatness)**
- $\exp(2\psi(r)) \neq 0 \ \forall \ r$ **(no horizons)**

Examples of $b(r)$ and $\psi(r)$:

- $b(r) = b_0$ **and** $\psi(r) = 0$ **(spatial Schwarzschild)**
- $b(r) = b_0$ **and** $\exp(2\psi) = \left(\kappa + \lambda\sqrt{1 - \frac{b_0}{r}}\right)^2$

(modified Schwarzschild, $R = 0$ solution)

ND, SK, SM, MV, PRD 65, 064004 (2002)

Einstein field equations and EC violations

• T_{ij} **in an orthonormal basis :**
$\rightarrow T_{00} = \rho$, $T_{11} = p_r$, $T_{22} = T_{33} = p_t$
• **Einstein field equations :**

$$\boxed{\begin{array}{c} \rho = \frac{1}{8\pi}\frac{b'}{r^2}\ ;\ p_r = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left\{1 - \frac{b}{r}\right\}\frac{\phi'}{r}\right] \\ p_t = \frac{1}{8\pi}\left\{1 - \frac{b}{r}\right\}\left[\phi'' + \phi'\left(\phi' + \frac{1}{r}\right)\right] \\ -\frac{1}{2}\left(\frac{b}{r}\right)'\left(\phi' + \frac{1}{r}\right) \end{array}}$$

• **WEC, NEC Violations**

$\rightarrow$ **Local EC imply** $\rho + p_r = \frac{b'r - b}{r^3} \geq 0$ **. Other constraints like** $\rho + p_t \geq 0$ **and/or** $\rho \geq 0$ **also exist.**

If the geometry is a wormhole are the EC violated? How?

Embedding of $t = const.$, $\theta = \frac{\pi}{2}$ slice in R^3

- Line element in R^3 in cylindrical coordinates

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2$$

- Take section $z = z(r)$ (profile function)

- Compare with the 2D section of the 4D line element

$$\left(\frac{dz}{dr}\right)^2 = \frac{b}{r-b}$$

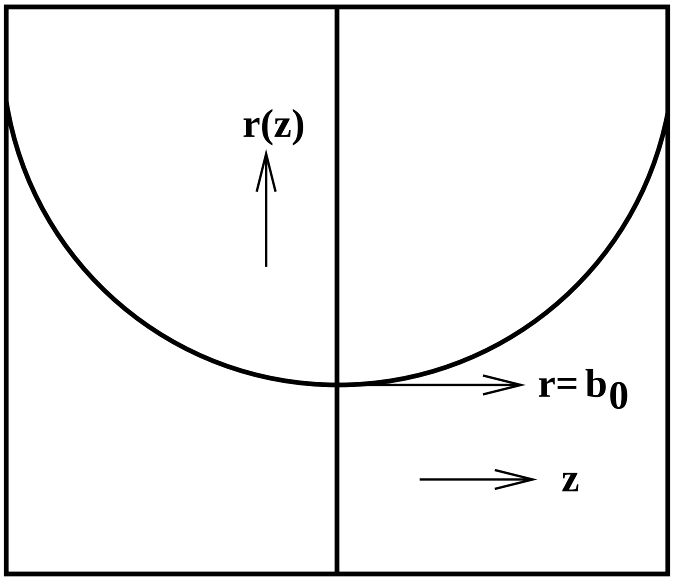
- For wormholes $r(z)$ has a minimum $z = 0$, $r = b_0$

$\rightarrow \frac{d^2r}{dz^2} > 0$ at $r = b_0$ $\rightarrow$ $b'r - b < 0$

$\rightarrow$ Recall that if EC holds then $b'r - b > 0$

- Thus, if wormholes exist $\rightarrow$ EC violated

[Morris–Thorne theorem (1988)]



• One may also think of a wormhole as

A DEFOCUSING LENS

Light rays from infinity converge to the throat But they do NOT get focused

Subsequently they diverge as they move away from the throat to the other asymptotic region

Expansion θ never reaches minus infinity

• Defocusing at the throat $\rightarrow$ negative energy density

Averaged Null Energy Condition (ANEC)

$$\boxed{\int \left[\rho + p_r\right] e^{-2\psi} d\lambda = -\frac{1}{4\pi} \int \frac{1}{r^2} e^{-\psi} \sqrt{1 - \frac{b(r)}{r}} dr}$$

• This is a line integral and does not tell us about the amount of negative energy!

Measure violations using‘volume integrals’

$$\boxed{\int \rho dV; \int \left[\rho + p_r\right] dV; \int \left[\rho + p_t\right] dV}$$

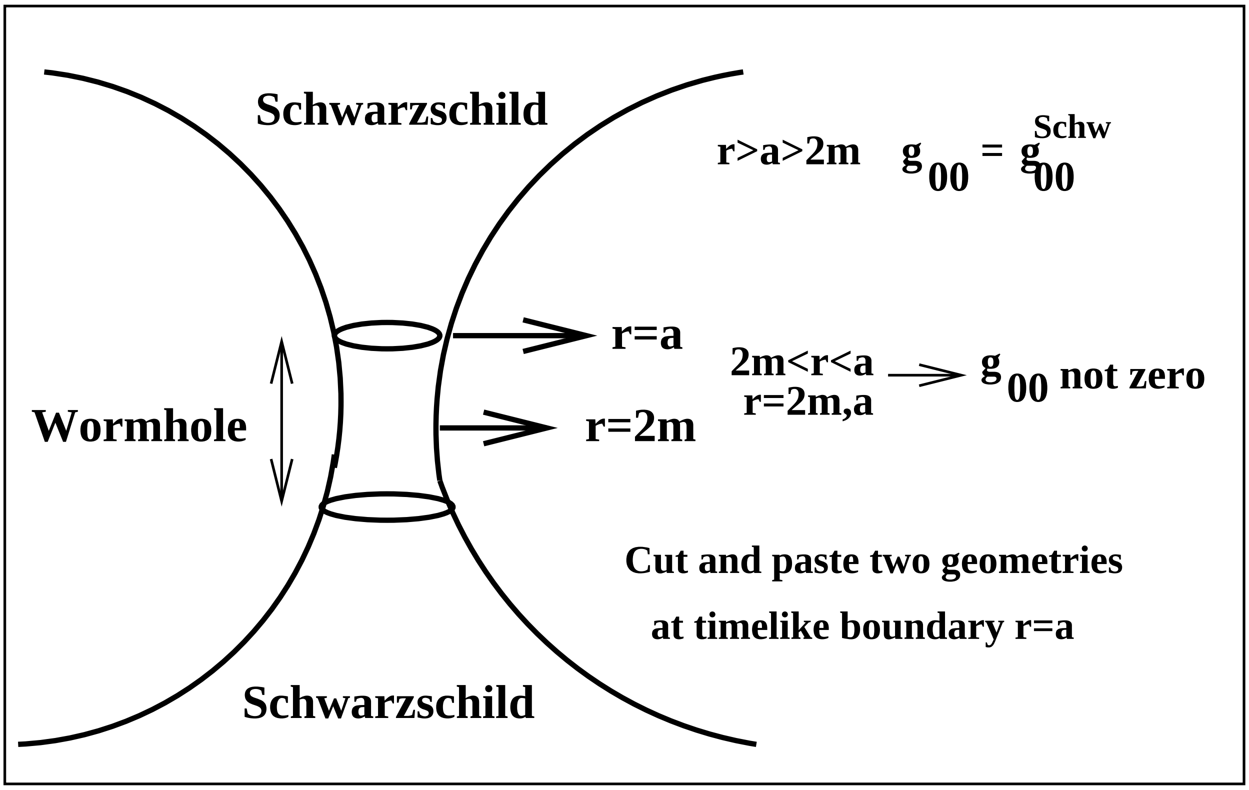
$\rightarrow$ Integration measure : $r^2 dr$ or $\sqrt{g} dr$

• To make wormholes with small amounts of EC violations use :

(i) Cut and paste different geometries

(ii) Smooth geometries with one metric

Cut and paste examples



- **The ANEC integral is always negative :**

$$I < -\frac{1}{2\pi}\int_a^\infty \frac{dr}{r^2} = -\frac{1}{2\pi a}$$

- **In general (using r^2dr measure) :**

$$\int [\rho + p_r]\, dV = -\int_{r_0}^{\infty} (1-b') \left[\ln\left(\frac{\exp[2\phi]}{1-b/r}\right)\right] dr.$$

- **If** $\rho = 0$**, the volume integral is** $\int p_r dV$**.**

$$\boxed{\int p_r \, dV = -\int_{r_0}^{a} \ln \left[\frac{\exp[2\phi]}{1-2m/r}\right] \, dr.}$$

Now $\int_{r_0}^{a} \ln \left[\frac{\exp[2\phi]}{1-2m/r}\right] dr < \int_{r_0}^{a} \ln \left[\frac{\exp[2\phi_{\max}]}{1-2m/r}\right] dr.$**Thus**

$$\boxed{\int p_r dV > -(a-2m) \ln \left[\frac{\exp[2\phi_{\max}]}{1-2m/a}\right] - 2\,m\,\ln\left(\frac{a}{2m}\right).}$$

- **Lower bound in terms of** $\phi_{\max}$ **and 'a'**

$$\boxed{\int p_r dV < -(a-2m) \ln \left[\frac{\exp[2\phi_{\min}]}{1-2m/a}\right] - 2\,m\,\ln\left(\frac{a}{2m}\right).}$$

- **Upper bound in terms of** $\phi_{\min}$ **and 'a'**

- **Choose** $\phi_{\max}$**,** $\phi_{\min}$ **, take limit** $a \to 2m$ **:**

$$\boxed{\int p_r dV \to 0}$$

• **Consider a sequence of traversable wormholes with suitably chosen a and $\phi(r)$ [and $b(r) = 2m$]**

• **Cut and paste them sequentially, thereby reducing the amount of negative energy density by choice of geometry**

• **Extrinsic curvatures to be matched across timelike boundaries**

• **Example:** $b(r) = 2m$ **(**$b_0 \leq r \leq \infty$**)**

$$b_0 \leq r \leq a : \; g_{00} = \left(\kappa + \lambda\sqrt{1 - \frac{2m}{r}}\right)^2$$

$$a < r \leq \infty : \; g_{00} = g_{00}^{Schw}, \; b(r) = 2m$$

Extrinsic curvature $\neq 0$ **at** $r = a$ **boundary**

• **Some smooth examples have also been worked out**

CONCLUSIONS

- We can quantify the amount of violation through spatial volume integrals.

- Qualitative results are unchanged for both the (i) $\sqrt{g}dr$ (ii) r^2dr measure. Actual amount calculations should use (i)

- ANEC is not the right quantifier because it is a line integral and does not yield the right dimensions of energy

- Wormholes can be designed at will with arbitrarily small amounts of violation of negative energy conditions

- The amount of violation is compatible with (i) quantum inequality restrictions (ii) traversability (Kuhfittig, 2003)

QUESTIONS

• Energy conditions can be violated, but the total amount of violation can approach zero from the negative side.

$\rightarrow$ Is the total amount more important that Local EC or Averaged EC?

$\rightarrow$ Our answer is YES. Our definition of VIOLATION is through the AMOUNT and NOT through the local/averaged EC.

• Is there a connection of this definition of violation with the Singularity Theorems which assume local/averaged EC?

• More examples need to be worked out to understand the importance of the volume integrals