Canonical quantum gravity

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1. Introduction: (recent) historical results
2. Thiemann's Hamiltonian constraint.
3. A recent development: Consistent discretizations.

In this talk I would like to review developments in canonical quantum gravity since the last ICGC.

I will not cover related development in path-integral quantum gravity (spin foams) nor cosmological applications (which will be covered by Martin Bojowald). In a separate talk I commented on possible experimental tests of quantum gravity (!).

There are other results I won't be able to cover for reasons of time
-Progress in the construction of a semi-classical picture
(Thiemann et. Al., Ashtekar et al., Varadarajan)
-Comments on the use of real Ashtekar variables (Samuel).
-The Kodama state (Freidel and Smolin).

I will start by giving some background reviewing results between 1985 and the last ICGC.

Some historical notes:

In 1985 Ashtekar noted that one could use a set of triads and a (complex) SO(3) connection to describe canonical quantum gravity. The variables allowed to view gravity as a sub-sector of the phase space of Yang-Mills theory, and made all constraint equations polynomial, and very attractive.

$$\tilde{E}^a{}_i, A_a{}^i \qquad \tilde{\tilde{g}}^{ab} = \tilde{E}^a{}_i \tilde{E}^b{}_i$$

$$D_a \tilde{E}^{ai} = 0$$

$$\tilde{E}^{ai} F_{ab}{}^i = 0$$

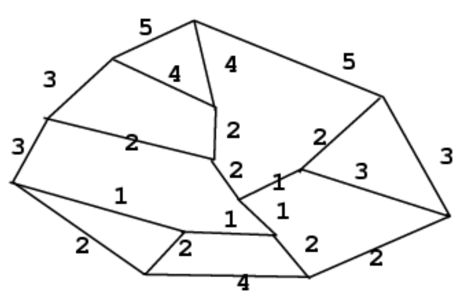
$$\tilde{E}^{ai} \tilde{E}^{bj} F_{ab}{}^k \epsilon_{ijk} = 0$$

The new variables changed the perspective on how to quantize the theory. Now the natural thing was to consider wavefunctions of a connection, like in Yang-Mills theories, $\Psi[A]$.

Rovelli and Smolin suggested in 1988 that one could use as a basis for such functions traces of holonomies, like Gambini and Trias had suggested in the early 80’s for Yang-Mills theories. The resulting representation is called the loop representation. $\Psi[\gamma]$, with γ a loop.

The loop representation not only automatically solved the Gauss law constraint, but if one considered diffeomorphism invariant functions of loops (knot invariants) one also solved the diffeomorphism constraint.

An important technical issue is that the basis of loops is overcomplete. Rovelli and Smolin noted in 1995 that one could find an easy way of labeling the independent linear combinations of elements of the basis using spin networks, an idea of Roger Penrose from the 60's.



Ashtekar and Isham, with further developments by Ashtekar and Lewandowski, had introduced a formal mathematical measure that allowed the integration of diffeomorphism invariant functions of a connection $\Psi[A]$. In terms of spin network states the measure acquires a very simple form. Essentially, states based on different networks are orthogonal.

Rovelli and Smolin noted in 1996 that on the Hilbert space of spin networks endowed with the Asthekar-Lewandowski measure, the area of a surface and the volume of a region, as quantum operators had discrete spectrum.

This led to the suggestion by Krasnov, Rovelli and Smolin that the entropy of a black hole could be viewed as counting the discrete number of degrees of freedom associated with the area of the horizon. Detailed calculations by Ashtekar, Baez, Corichi and Lewandowski confirmed that the entropy is proportional to the area of the hole. This is a fully dynamical calculation.

Implementing the Hamiltonian constraint as a quantum operator remained elusive until in 1997 Thiemann found a series of classical identities that allowed to promote the (apparently non-polynomial) single-densitized Hamiltonian constraint to a quantum operator. He did so by discretizing the theory on a lattice and remarkably, the lattice spacing drops out of the calculation if one operates on diffeomorphism invariant states.

$$\frac{\tilde{E}^a{}_i \tilde{E}^b{}_j e^{ijk}}{\sqrt{\tilde{E}^a{}_i \tilde{E}^b{}_j \tilde{E}^c{}_k e^{ijk} e_{abc}}} = 2 e^{abc}\{A_c{}^k, V\} \qquad \tilde{H} = -2\mathrm{Tr}(F_{ab}\{A_c, V\}) e^{abc}$$

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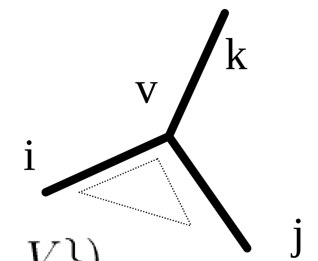


$$H^E_\Delta[N] := -\frac{2}{3} N_v \epsilon^{ijk} \mathrm{tr}\left(h_{\alpha_{ij}(\Delta)} h_{s_k(\Delta)} \{h^{-1}_{s_k(\Delta)}, V\}\right)$$

Thiemann’s Hamiltonian has features that are not well understood yet. In particular, it appears to have too many solutions (acting on a bra state that does not include “exceptional vertices” but otherwise is unrestricted, the action vanishes).

Moreover, the Hamiltonian, since its action is only defined on diffeomorphism invariant states, has an Abelian constraint algebra. Attempts to generalize the action to “slightly diffeomorphism invariant” States failed to yield a non-Abelian algebra (Lewandowski and Marolf 1990).

This led us, with Rodolfo Gambini to try to look in more detail at what is going on in Thiemann’s construction.

Continuum Classical Theory in terms of E,A

Discrete classical theory in terms of holonomies and volumes on a triangulation

What do we know about this classical theory?

Quantum theory

We know precious little, as we actually always do when we handle discretizatons of theories… Our aim is to understand these kinds of theories. I will devote the remainder of the talk to this point.

It should be noticed that Thiemann quantizes while at the same time taking a limit to the space of solutions of the diffeomorphism contraint.

Consistent discretizations: Key idea

R. Gambini, JP, Phys. Rev. Lett 90 021301 (2003)

If one discretizes the Einstein equations one gets a set of algebraic equations that is inconsistent (cannot be solved simultaneously).

Example: Take the 3+1 form of the equations. One has twelve evolution equations and four constraints, yet only twelve variables to solve for (extrinsic curvature and three-metric).

Proposal: To make the equations consistent, consider the lapse and shift as dynamical variables. Then one has 16 equations for 16 unknowns.

The resulting theory is different from GR, yet it will generically include solutions that approximate continuum GR very well.

Quantization:

We have developed the tools to treat these kinds of theories canonically. Since time is discrete, it does not make sense to have a Hamiltonian. Time evolution is implemented through a finite canonical transformation that takes the system from n to n+1.

Quantization is done by implementing the canonical transformation as a unitary operator.

C. Di Bartolo, R. Gambini, JP, Class. Quan. Grav. 19, 5275 (2002)

Canonical treatment of discrete theories:

We start with a discrete action, $$S = \sum_{i=1}^{N} L(q_i, q_{i+1})$$

with Lagrange equations $$\frac{\partial S}{\partial q_n} = \frac{\partial L(q_{n-1}, q_n)}{\partial q_n} + \frac{\partial L(q_n, q_{n+1})}{\partial q_n} = 0.$$

We introuduce the function F which will be the generator of discrete time evolution. The evolution is implemented through a type 1 canonical transformation with F its generating function,

$$F_1(q_n, q_{n+1}) \equiv -L(q_n, q_{n+1})$$
$$p_n = \frac{\partial F_1}{\partial q_n} = \{F_1, p_n\}$$
$$p_{n+1} = -\frac{\partial F_1}{\partial q_{n+1}} = -\{F_1, p_{n+1}\}.$$

This transformation maps the pair (q_n,p_n) to (q_{n+1},p_{n+1}) preserving the Poisson bracket structure.

To quantize one of these systems we pick a polarization for the wavefunctions, for instance, $\Psi(q_i)$, we then need to find a unitary transformation that corresponds to the canonical transformation F, in this case it is,

$$p_i = U p_{i-1} U^\dagger,\ q_i = U q_{i-1} U^\dagger$$

$$U = \exp\left(i \frac{V(q_{i-1})\epsilon}{\hbar}\right) \exp\left(i \frac{p_{i-1}^2 \epsilon}{2m\hbar}\right).$$

The Hamiltonian will not be conserved under evolution, as we discussed. However, the above expression for U suggests we can write $U=\exp(i\varepsilon H_{eff}/h)$, where H_{eff} is easily constructed with the Baker-Campbell-Hausdorff formula. One can immediately find a classical counterpart for this conserved quantity, it involves infinitely many terms.

Constrained systems:

We start from the Lagrangian,

$$L(n, n+1) = p_n \left(q_{n+1} - q_n\right) - \epsilon H\left(q_n, p_n\right) - \lambda_{nB}\, \phi^B\left(q_n, p_n\right)$$

We again define a generating function for discrete time evolution, F=-L. The canonical transformation yields,

$$P^q_{n+1} = \frac{\partial L(n, n+1)}{\partial q_{n+1}} = p_n$$
$$P^p_{n+1} = \frac{\partial L(n, n+1)}{\partial p_{n+1}} = 0$$
$$P^{\lambda_B}_{n+1} = \frac{\partial L(n, n+1)}{\partial \lambda_{(n+1)B}} = 0.$$

From these we can make explicit the equations of motion of the system,

$$p_n - p_{n-1} = -\epsilon \frac{\partial H(q_n, p_n)}{\partial q_n} - \lambda_{nB} \frac{\partial \phi^B(q_n, p_n)}{\partial q_n}$$
$$q_{n+1} - q_n = \epsilon \frac{\partial H(q_n, p_n)}{\partial p_n} + \lambda_{nB} \frac{\partial \phi^B(q_n, p_n)}{\partial p_n}$$
$$\phi^B(q_n, p_n) = 0.$$

$$\phi^B(q_{n+1}, P^q_{n+1}, \lambda_{nB}) = 0, \longrightarrow \lambda_{nB} = \lambda_{nB}(q_{n+1}, P^q_{n+1}, v^\alpha)$$

The equations look superficially like the continuum ones, but they are considerably more complicated, for instance since $P^q_{n+1}=p_n$ constraints at different times do not have vanishing Poisson brackets (!). The solution consists in eliminating the constraints, this determines the Lagrange multipliers.

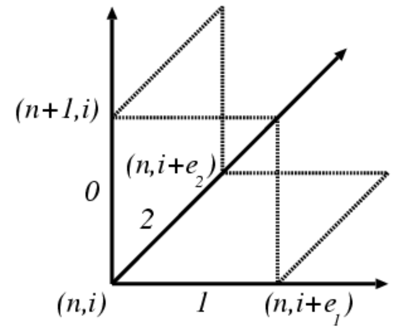
We have applied the framework to Yang-Mills theory, BF-theory and general relativity. In the case of BF theory, it provides the first direct lattice treatment ever constructed.

In the case of general relativity, although there are no obstructions in principle, working out the details in concrete situations is what will really be important. For instance: the equations that determine the Lagrange multipliers are non-linear coupled algebraic equations. Are their solutions real? What about various possible “branches” (in the case of field theory situations, many branches per point).

The credibility of the whole approach will be built by studying situations of ever increasing complexity.

A topological field theory: BF

$$S = \int \mathrm{Tr}\,(B \wedge F) \qquad \begin{aligned} D_a E^a &= 0, \\ F_{ab} &= 0, \end{aligned}$$



$$L(n) = -i \sum_i \mathrm{Tr}\left[B^0_{n,i} h^1_{n,i} h^2_{n,i+e_1} \left(h^1_{n,i+e_2}\right)^{-1} \left(h^2_{n,i}\right)^{-1} + B^2_{n,i} V_{n,i} h^1_{n+1,i} \left(V_{n,i+e_1}\right)^{-1} \left(h^1_{n,i}\right)^{-1} \right.$$
$$\left. + B^1_{n,i} h^2_{n,i} V_{n,i+e_2} \left(h^2_{n+1,i}\right)^{-1} \left(V_{n,i}\right)^{-1}\right] + \sum_{k=1}^{2} \lambda^k_{n,i} \left(h^k_{n,i} \left(h^k_{n,i}\right)^{\dagger} - 1\right) + \mu_{n,i} \left(V_{n,i} V^{\dagger}_{n,i} - 1\right).$$

Fundamental variables: $h^k_{n,i},\ V_{n,i},\ B^k_{n,i},\ \lambda^k_{n,i}$ and $\mu^k_{n,i}$.

$$\begin{aligned}
\Pi^{h^1}_{n+1,i} &= -i \left(V_{n,i+e_1}\right)^{-1} \left(h^1_{n,i}\right)^{-1} B^2_{n,i} V_{n,i} \\
\Pi^{h^2}_{n+1,i} &= -i \left(V_{n,i+e_2}\right)^{-1} \left(h^2_{n,i}\right)^{-1} B^1_{n,i} V_{n,i} \\
\Pi^{V}_{n+1,i} &= 0 \\
\Pi^{B}_{n+1,i} &= 0 \\
\Pi^{\lambda^k}_{n+1,i} &= 0 \\
\Pi^{\mu}_{n+1,i} &= 0,
\end{aligned}$$

Canonical momenta,
Primary constraints

Equations of motion:

$$h^k_{n+1,j} = \nu^k_{n,j}$$
$$\Pi^1_{n,j} = ih^2_{n,j+e_1}\left(h^1_{n,j+e_2}\right)^{-1}\left(h^2_{n,j}\right)^{-1}B^0_{n,j} - \lambda^1_{n,j}\left(h^1_{n,j}\right)^{-1} - iV_{n,j+e_1}\left(\nu^1_{n,j}\right)^{-1}V^{-1}_{n,j}B^2_{n,j}$$
$$\Pi^2_{n,j} = -ih^1_{n,j+e_2}\left(h^2_{n,j+e_1}\right)^{-1}\left(h^1_{n,j}\right)^{-1}B^0_{n,j} - \lambda^2_{n,j}\left(h^2_{n,j}\right)^{-1} - iV_{n,j+e_2}\left(\nu^2_{n,j}\right)^{-1}V^{-1}_{n,j}B^1_{n,j}.$$

Preservation of the constraints,

$$\Pi^V_{n,j} = i\nu^1_{n,j}V^{-1}_{n,j+e_1}\left(h^1_{n,j}\right)^{-1}B^2_{n,j} - i\nu^2_{n,j}V^{-1}_{n,j+e_2}\left(h^2_{n,j}\right)^{-1}B^1_{n,j}$$
$$-i\left(\nu^1_{n,j-e_1}\right)^{-1}V^{-1}_{n,j-e_1}B^2_{n,j-e_1}h^1_{n,j-e_1}i\left(\nu^2_{n,j-e_2}\right)^{-1} + V^{-1}_{n,j-e_2}B^1_{n,j-e_2}h^2_{n,j-e_2} = 0$$
$$\Pi^{B^3}_{n,j} = i\mathrm{Tr}\left(h^{12}_{n,j}T^A\right)T^A = 0$$
$$\Pi^{B^2}_{n,j} = i\mathrm{Tr}\left(\nu^1_{n,j}V^{-1}_{n,j+e_1}\left(h^1_{n,j}\right)^{-1}T^AV_{n,j}\right)T^A = 0$$
$$\Pi^{B^1}_{n,j} = \mathrm{Tr}\left(\nu^2_{n,j}V^{-1}_{n,j+e_2}\left(h^2_{n,j}\right)^{-1}T^AV_{n,j}\right) = 0$$

Last two equations determine nu's, second equation will be F=0,
The first equation, substituting nu in it, and rewriting it in terms of,

$E^k_{n,j} = \Pi^k_{n,j}h^k_{n,j}$, one gets Gauss' law $E^2_{n,j} + E^1_{n,j} + E^2_{n,j-e_2} + E^1_{n,j-e_1} = 0.$

Quantum cosmology: R. Gambini, JP, CQG 20, 3341 (2003)

As a direct application, let us consider the quantization of a simple cosmology, the Friedmann model with cosmological constant and a (very massive) scalar field. In terms of Ashtekar variables,

$$L = E\dot{A} + \Pi\dot{\phi} - NE^2(-A^2 + \Lambda E + m^2\phi^2 E) \qquad N = E^{-3/2}\alpha$$

$$A = \exp(\sqrt{\Lambda + \phi^2}t) \qquad E = \exp(2\sqrt{\Lambda + m^2\phi^2}t)/(\Lambda + m^2\phi^2)$$

Discretizing:

$$\begin{aligned} L_n &= E_n\left(A_{n+1} - A_n\right) + \pi_n\left(\phi_{n+1} - \phi_n\right) \\ &\quad -N_n E_n^2\left(-A_n^2 + \Lambda E_n + m^2\phi^2\right) \end{aligned} \qquad F_1 = -L$$

E and its conjugate (=0) disappear from the canonical transformation. Conservation of Π^N=0 implies that the Hamiltonian constraint is conserved.

If one works out the discrete equations of motion and makes some substitutions, they all boil down to a single recursion relation,

$$A_{n-1}^2 - A_n^2 + 2\left(A_{n+1} - A_n\right) A_n = 0$$

Classically, this recursion relation determines the evolution of the universe given two values of A_n. The universe expands.

What happens to the singularity? The singularity would occur when E=0, that is A=mϕ. But if one runs the recursion relation backwards, one will only achieve such value for A if one fine tunes the initial data. **Generically therefore the singularity is avoided in the classical theory**.

For large n, the recursion implies, $A \to n^{2/3}, E \to n^{4/3} \;\; N \to n^{-3}$

Choosing $t = \epsilon n \quad E \to t^{4/3}$

Which coincides with the classical solution in the slicing $\alpha=t^{-1}$

Quantization: $\hat{P}\Psi(P,f) = P\Psi(P,f), \quad \hat{A}\Psi(P,f) = i\frac{d}{dP}\Psi(P,f)$

Reality conditions: $A^+ = A, \quad \mathrm{P}^+ = P$

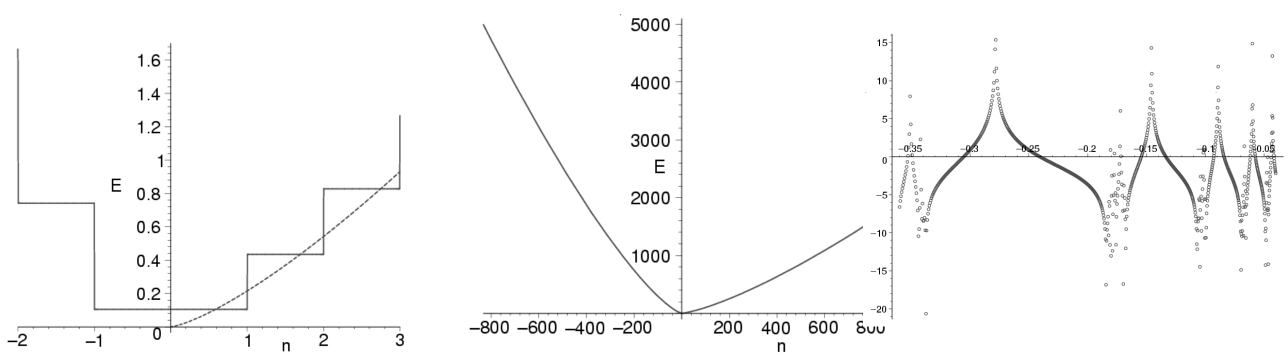
Inner product: $$<\Psi|\Phi> = \int_0^\infty dP \int_{-\infty}^\infty df\; e^{-P}\Psi^*(P,f)\Phi(P,f)$$

Implements reality conditions if $\Psi(0,\phi)=\Phi(0,\phi)=0$

As expected, the singularity is completely removed at a quantum level, since it occurs for only one fine-tuned value of initial classical conditions. At a quantum mechanical level such situation has probability zero.

The life of the cosmos: R. Gambini, JP, **IJMPD12, 1775 (2003)**

The discrete theory replaces the big bang with a tunneling into a new universe. In such tunneling the lattice spacing changes.



If one were to imagine a lattice gauge theory on such lattice, the change in the lattice spacing could be viewed as a change in the dressed values of fundamental physical constants. The change is random. This is a practical implementation of Smolin’s Darwinian cosmology “The life of the cosmos”.

Solution of the problem of time:

Since the discrete theory is constraint-free, it is free from many of the hard conceptual problems of GR.

For instance, one can implement the Page-Wootters relational quantum mechanics, in which all variables are promoted to operators and one of these quantum operators is considered a "clock" and one can make predictions about conditional probabilities.

The parameter "n" in the discretization is just an ontological bookkeeping tool, real "time" is introduced via a relational approach.

$$P_{\mathrm{cond}}(P^\phi = x|A = t) =$$
$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} \Psi^2[A{=}t, P^\phi{=}x, n]}{\sum_{n=0}^{N} \int_{-\infty}^{\infty} \Psi^2[A{=}t, P^\phi, n]\, dP^\phi}$$

The main objections to the Page-Wootters quantization were related to the fact that GR is a constrained theory. What should one use as observables to perform the construction? (cf. Discussion by Kuchar in his Problem of Time review).

In the discrete theory, since there are no constraints, there is no obstruction to implementing the Page-Wootters quantization.

R. Gambini, R. Porto, JP gr-qc/**0302064, gr-qc/0305098**

Possible experimental consequences:

The relational quantum mechanics of Page-Wootters is not equivalent to ordinary Schroedinger quantum mechanics. It approximates it well when there is a variable in the problem that can play well a role of a classical clock. It is very different in extreme situations when no classical clock is available.

Even in situations where it approximates well ordinary quantum mechanics, one should expect small departures. In particular if one considers a system with two levels, it will eventually lose coherence by an amount:

$$(w_1 - w_2)^2 t_{\text{Planck}} \ t_{\text{tof}}$$

Where omega is the system frequency and t_{tof} is the "time of flight". This could have observable consequences for for neutron interferometry and "macroscopic" quantum systems (ie. Bose-Einstein condensates).

Black hole information paradox:

The usual black hole information paradox refers to taking a pure state, collapsing it into a black hole and letting it evaporate completely. Since The final state is thermal it implies that a pure state evolved into a a mixed state. This is not allowed in ordinary quantum mechanics.

But we just showed that in relational quantum mechanics, pure states evolve naturally into mixed states. Therefore this would provide a way to avoid the paradox, provided the decoherence mechanism is faster than black hole evaporation. This in principle appears not possible, Since the decoherence effect is minute. However, black hole evaporation is also slow. If one makes the black hole smaller, evaporation happens faster, but also the decoherence of relational quantum gravity since the energy fluctuations are larger!

It should be noted that our proposal has some similarity with Hawking's \$-matrix proposal (decoherence via interaction with the space-time foam), but it does not have the problem of lack of conservation of energy. It leads to a Lindblad type of evolution,

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] - \mathcal{D}(\rho), \quad \mathcal{D}(\rho) = \sum_n [D_n, [D_n, \rho]], \quad D_n = D_n^\dagger, \quad [D_n, H] = 0,$$

A quick calculation shows that the decoherence wins over evaporation for black holes of masses larger than 600 Planck mass. Since for smaller black holes one cannot really speak about evaporation in the traditional Hawking sense, the information paradox never arises!

Other applications:

The consistent discretizations could offer promise for **numerical relativity**. Since the discretizations can be understood canonically, this provides a tool for the construction of exact conserved quantities for the discrete theory. The construction of conserved quantities is a usual numerical analysis tool to prove that a discretization scheme is stable.

See for instance
G. Calabrese, L. Lehner, D. Neilsen, JP, O. Reula, O. Sarbach, M. Tiglio, gr-qc/0302072

We have recently shown that for linearized general relativity our approach leads naturally to a “mimetic” discretization scheme.

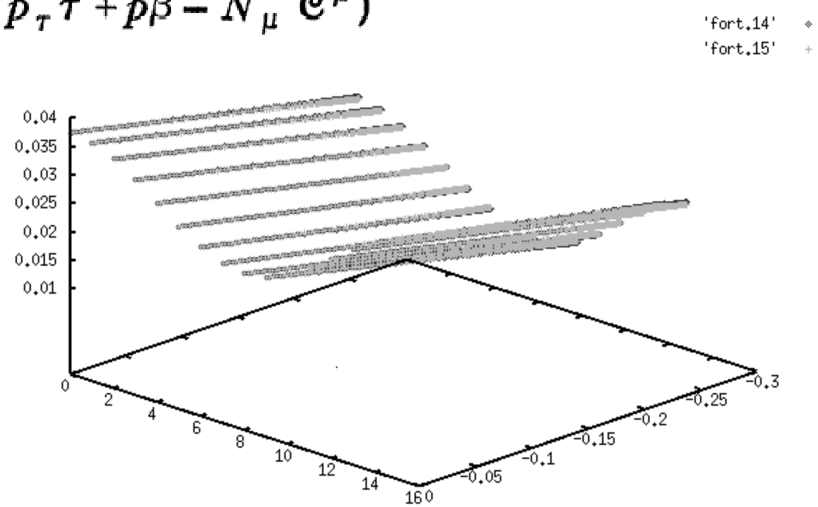
Coming soon: the Gowdy cosmology

We are studying the Gowdy cosmologies without a complete gauge fixing. One has a Hamiltonian and one component of the momentum constraint. One solves for the lapse and one component of the shift.

$$dl^2 = e^{-\tau-(\lambda/2)}\, d\theta^2 + e^{2\tau}(e^{\beta}d\sigma^2 + e^{-\beta}d\delta^2)$$

$$2\pi I = \int\int dt\, d\theta(p_\lambda \dot{\lambda} + p_\tau \dot{\tau} + p\dot{\beta} - \overline{N}_\mu\, \overline{\mathcal{C}}^\mu)$$

β vs τ



Summary

- The new consistent discretization scheme offers several attractive possibilities for implementing dynamics in classical and quantum gravity.
- It appears to work well in simple settings, with progress even in the field theoretical cases (Gowdy).
- The main challenge is to show it will work in more elaborate settings, and that it leads to (more or less) unique predictions.