Integrability in 2d gravity

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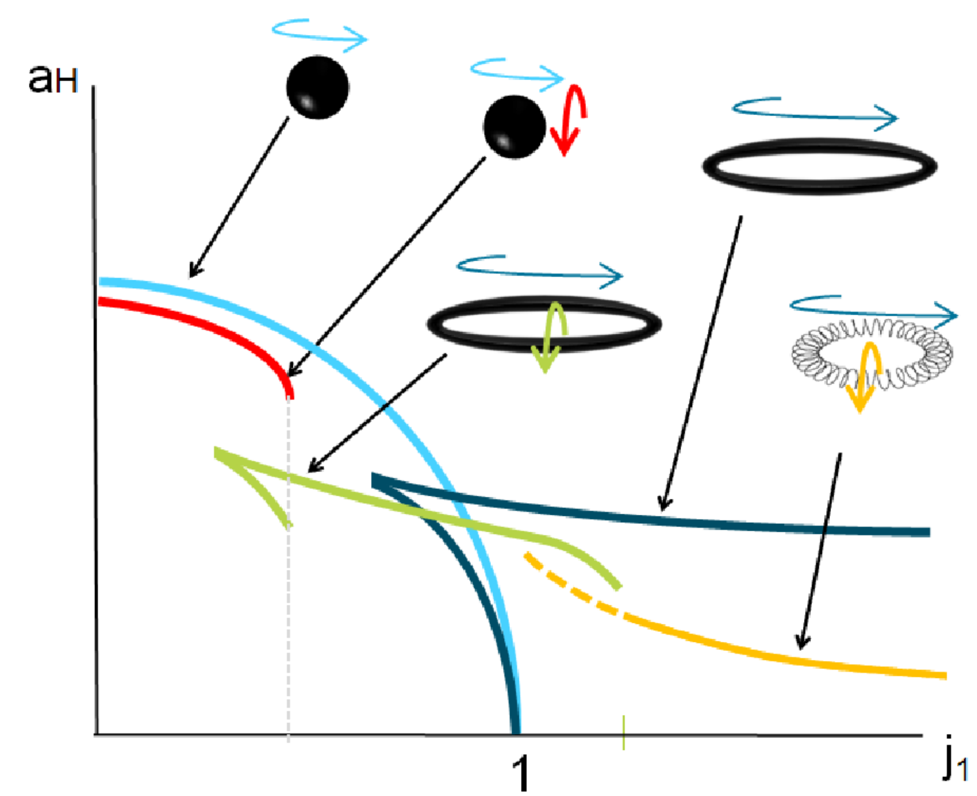
Pursuit of exact solutions

- Exact solution are precious. They are hard to obtain.
- Gravity in higher dimensions has attracted much attention in the last decade.
- Gravity in higher dimensions is much richer…and much harder to find exact solution
    - AdS/CFT
    - String theory
    - TeV-scale gravity

The black hole bestiary

- How much do we know about axisymmetric solutions in higher dimensions?
- In 4d Kerr-Newman solution exhaust the space
- In 5d much progress has been achieved over the last decade
- For 6d or higher dimensions, solutions are scarcer…



Solution generating techniques

- “It often happens that when one is trying to solve an equation that an algorithm exists for constructing new solutions from a given one.” **[Wald 1984]**
- The first solution generating technique applied to gravity was due to Ehlers and later developed further by Geroch
- A class of solution-generating techniques is provided by the hidden symmetries of dimensionally reduced theory to 3d
- The so-called Backlund transformations were developed in late 70s
- The inverse scattering technique appeared [Belinski and Zakharov] 1978
- Various innovative improvements have been made recently, making these techniques promising and powerful for the future

Solution generating techniques

- Various (super)-gravity theories in D-dimensions when reduced to 2d, reduce to 2d gravity coupled to non-linear sigma model
- Such 2d models are known to be integrable
- Integrability of these models **had not** been used as a solution generating techniques
- Notable exceptions are 4d vacuum gravity and 4d Einstein Maxwell theory [1980s—1990s]

Goal

- Explore integrability of various supergravities and implement inverse scattering methods. Make them practical.

Motivation

- Dimensionally reduced gravity theories have large U-duality groups
    - 4d gravity in 3d: SL(2,R);
    - 11d supergravity in 3d: E8(8)
- These symmetries have been used to study black holes:
    - uniqueness results
    - charged black holes
    - BPS and non-BPS from duality orbits
- These symmetries are just a tip of an iceberg. In many situations infinite dimensional extensions are available.
    - Black holes/fuzzballs in 4d (5d) have 2 (3) commuting Killing vectors. Thus we have access to symmetries of theories reduced to 2d, which are infinite dimensional.

Motivation

- We want to understand and make use of these symmetries.
- Using these symmetries one can address many difficult problems
    - How to describe thermal excitations over multi-center supersymmetric bound states?
    - To construct a bigger family of non-supersymmetric fuzzballs?
- No known techniques to construct such solutions and address such problems.
    - Our formalism is sufficient to address these problems; though the achievements have been only modest so far.

Plan

- Review of the inverse scattering method
- Review of the hidden symmetries
- Geroch Group; our formalism
- Vision and comments

Inverse scattering

Review: Inverse Scattering Method
Canonical coordinates

- Consider stationary axisymmetric solutions of Einstein's equations
- Assume D-2 commuting Killing vectors
- Weyl canonical coordinates; metric components only depend on (ρ, z)

$$ds^2 = \sum_{i,j=1}^{N} G_{ij} dx^i dx^j + e^{2\nu} \left[d\rho^2 + dz^2 \right]$$

- Einstein's equation can be divided in two groups
  - For G_{ij} $\quad \partial_\rho U + \partial_z V = 0, \quad \text{where} \quad U = \rho(\partial_\rho G)G^{-1}, \quad V = \rho(\partial_z G)G^{-1}$
  - For ν $\quad \partial_\rho \nu = -\frac{1}{2\rho} + \frac{1}{8\rho}\mathrm{tr}(U^2 - V^2), \quad \partial_z \nu = \frac{1}{4\rho}\mathrm{tr}(UV)$
- Integrability condition $\partial_\rho \partial_z \nu = \partial_z \partial_\rho \nu$ is automatically satisfied

Review: Inverse Scattering Method
Static solutions

- It is easy to construct static solutions [Weyl solutions]

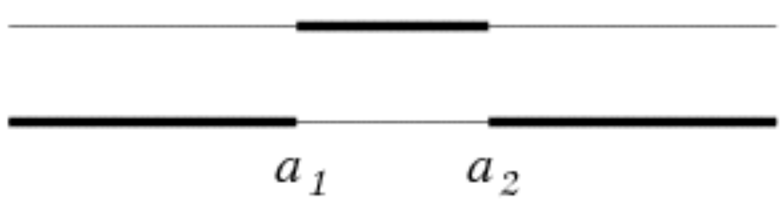
$$G = \mathrm{diag}\{-e^{2U_0}, e^{2U_1}, e^{2U_2}, \dots\}$$

- The functions Us are simply Harmonic functions in 3d [auxiliary] flat space subject to a constraint

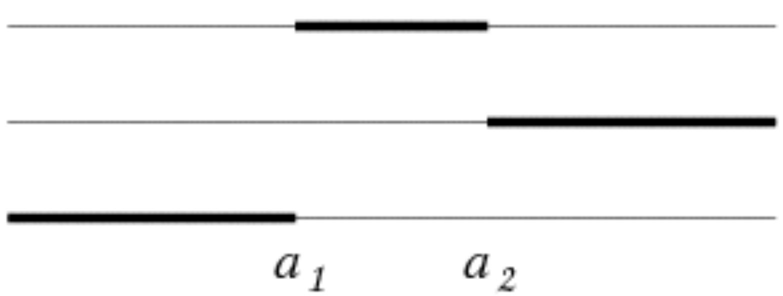
$$\nabla^2 U_i = 0, \qquad \sum_{i=0}^{D-3} U_i = \log \rho$$

- In order to solve these equations, boundary conditions should be specified on the z-axis. Most interesting solutions occur with zero-thickness rod sources.

Review: Inverse Scattering Method
Static solutions



(a)



(b)

- For infinite rod $e^{2U_i} = \rho^2$
- For semi-infinite rods running to the right $e^{2U_i} = \mu_k$
- For semi-infinite rods running to the left $e^{2U_i} = \rho^2/\mu_k \equiv \bar{\mu}_k$
- For finite rod $e^{2U_i} = \mu_{k-1}/\mu_k$
- Solitons and anti-solitons

$$\mu_k = \sqrt{\rho^2 + (z - a_k)^2} - (z - a_k), \qquad \bar{\mu}_k = -\sqrt{\rho^2 + (z - a_k)^2} - (z - a_k)$$

Review: Inverse Scattering Method
Stationary solutions

- Static vacuum solutions of Einstein equations with D-2 commuting Killing vectors are completely determined by rod-like sources. [Emparan Reall]

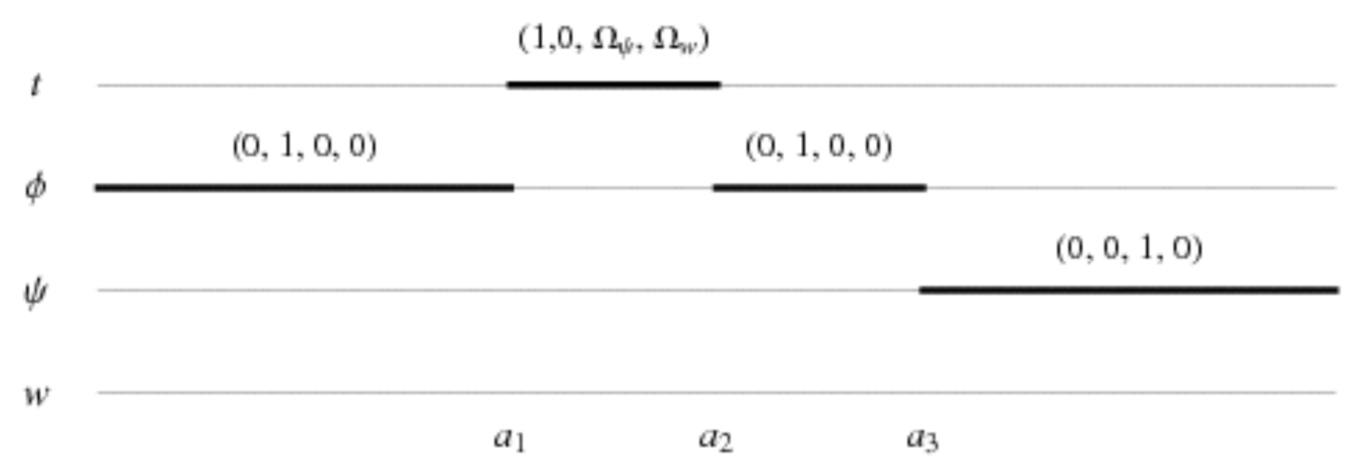
- Rod structure classification can be extended for stationary black holes. The rods now acquire directions.

- Metric is no longer diagonal $G(\rho = 0, z)v_k = 0 \quad \text{for} \quad z \in [a_{k-1}, a_k]$

- These "rod vectors" encode information about the rotations, etc

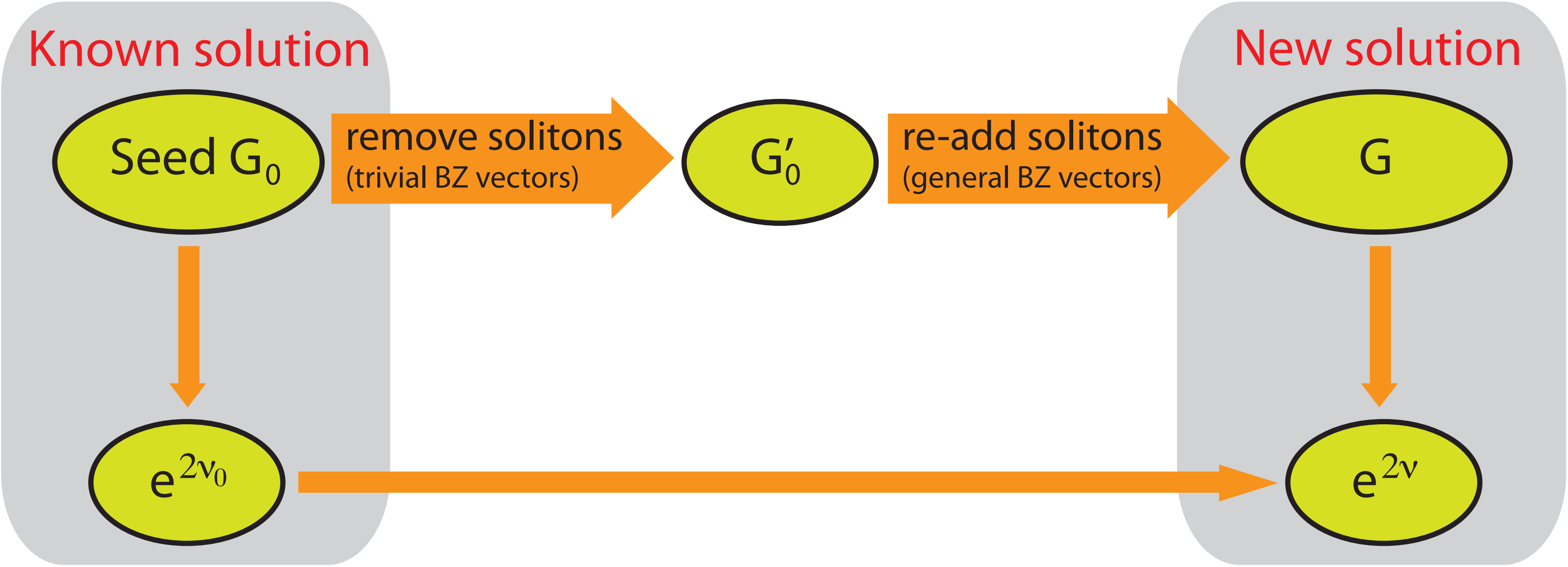
A typical rod diagram

[Harmark 2004]



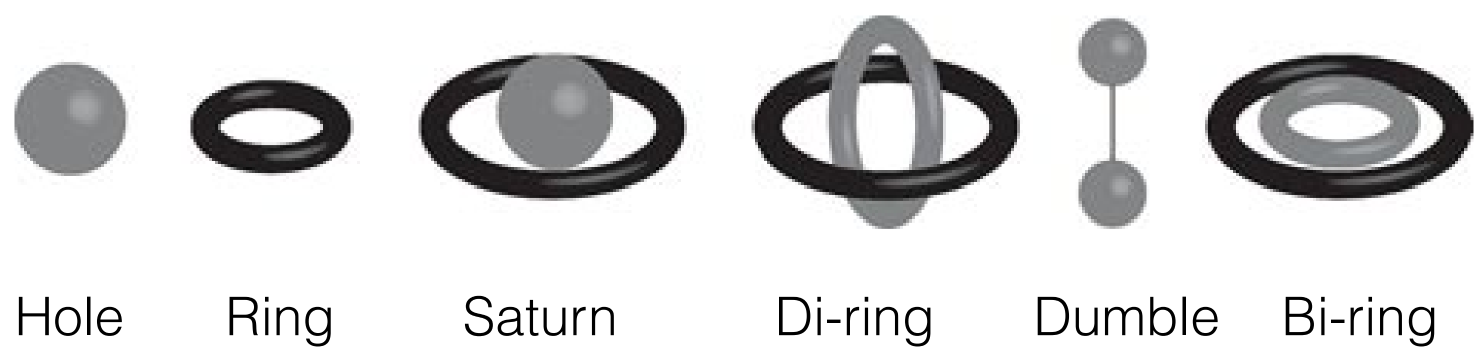
Belinski Zakharov Method

- BZ method starts by replacing non-linear equations by a system of linear (Lax) equations
- The Lax system encodes the integrability structure and allows for a dressing method.



[Pomeransky 2005]

Some exact solutions, using BZ method



- Many many exact solutions have been constructed
- Techniques have been most useful in 5d gravity
- Some applications to other settings also, relatively few

[many authors]

Hidden symmetries

Review: Hidden symmetries

- Start with a higher dimensional gravity theory
- Perform Kaluza Klein reduction over commuting Killing vectors
- The effect is most dramatic when reduced to three dimensions
- As in 3d dimensions, all vectors can be dualised to scalars

Review: Hidden symmetries

These scalars usually come in a coset space.

$$\mathcal{L} = R \star_3 1 - \frac{1}{8}\mathrm{tr}\left[(M^{-1}dM) \wedge \star_3 (M^{-1}dM)\right]$$

4d gravity (Ehlers)	$\frac{\mathrm{SL}(2,R)}{\mathrm{SO}(1,1)}$
Vacuum 5d gravity	$\frac{\mathrm{SL}(3,R)}{\mathrm{SO}(2,1)}$
Minimal 5d supergravity	$\frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(2,2)}$
STU supergravity	$\frac{\mathrm{SO}(4,4)}{\mathrm{SO}(2,2)\times\mathrm{SO}(2,2)}$

Review: Hidden symmetries

- These hidden symmetries can be used to construct solutions of of theory

- The group structure helps in organising solutions, allows to explore the physics and the structure of the solutions/theory

- Prolific subject; many results; identities; solutions;

- A lot of richness; way of doing the reduction; denominator groups

Review: Hidden symmetries: key results

- 5d charged rotating black holes; 4d charged rotating black holes [Cvetic Youm, Sen, Rasheed, Larsen]
- Most general black string in 5d [Compere, de Buyl, Jamsin, A.V.]
- Subtracted geometries [A.V.]
- Most general black hole in 4d [Compere, Chow]
- Classification of BPS and non-BPS multi-centre solutions [Bossard et al, A.V. et al]

Our formalism

Combines integrability with hidden symmetries

series of 4 papers with Kleinschmidt, Katsimpouri, Chakrabarty

Geroch Group

- Geroch group is the symmetry of 4d gravity reduced to 2d.
- 2d gravity has an infinite dimensional symmetry; it is integrable; symmetry group is called the Geroch Group.
- It is affine $SL(2, R)$. [non-compact Kac-Moody group]

Generalized Geroch Group

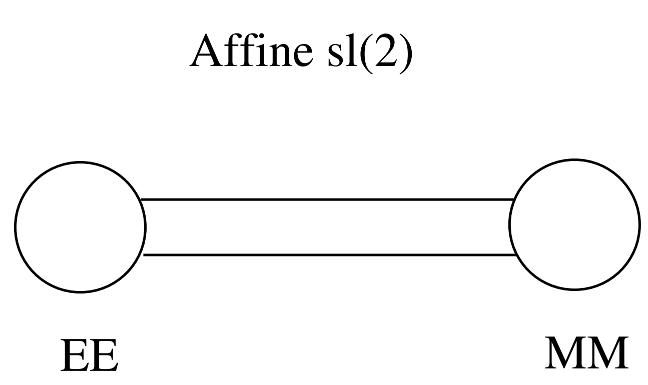
- These considerations have natural generalization to other settings
  - 5d gravity has SL(3,R) affine symmetry
  - 4d Einstein Maxwell: SU(2,1)
  - 5d Minimal supergravity: G2(2)
  - STU model: SO(4,4)

Ehlers and Matzner-Misner

- A natural way to think about affine SL(2,R) is to think about two non-commuting SL(2,R)s
- Ehlers: Upon reducing 4d gravity to 3d, we get SL(2,R)/SO(1,1) scalar symmetry
- Matzner-Misner: Upon reducing 4d gravity to 2d, we get SL(2,R)/SO(2) scalar symmetry

Dynkin Diagram

These two symmetries ***do not*** commute. They form affine SL(2,R).



Geroch 1971, Julia 1980

Geroch Group Matrices

- A matrix in the Geroch group, with suitable analyticity properties corresponds to a solution of Einstein equations

- In order to construct the solution explicitly requires one to solve a Riemann Hilbert problem.

- In a sequence of papers we have solved this problem for general affine group G. Kleinschmidt, Katsimpouri, AV.

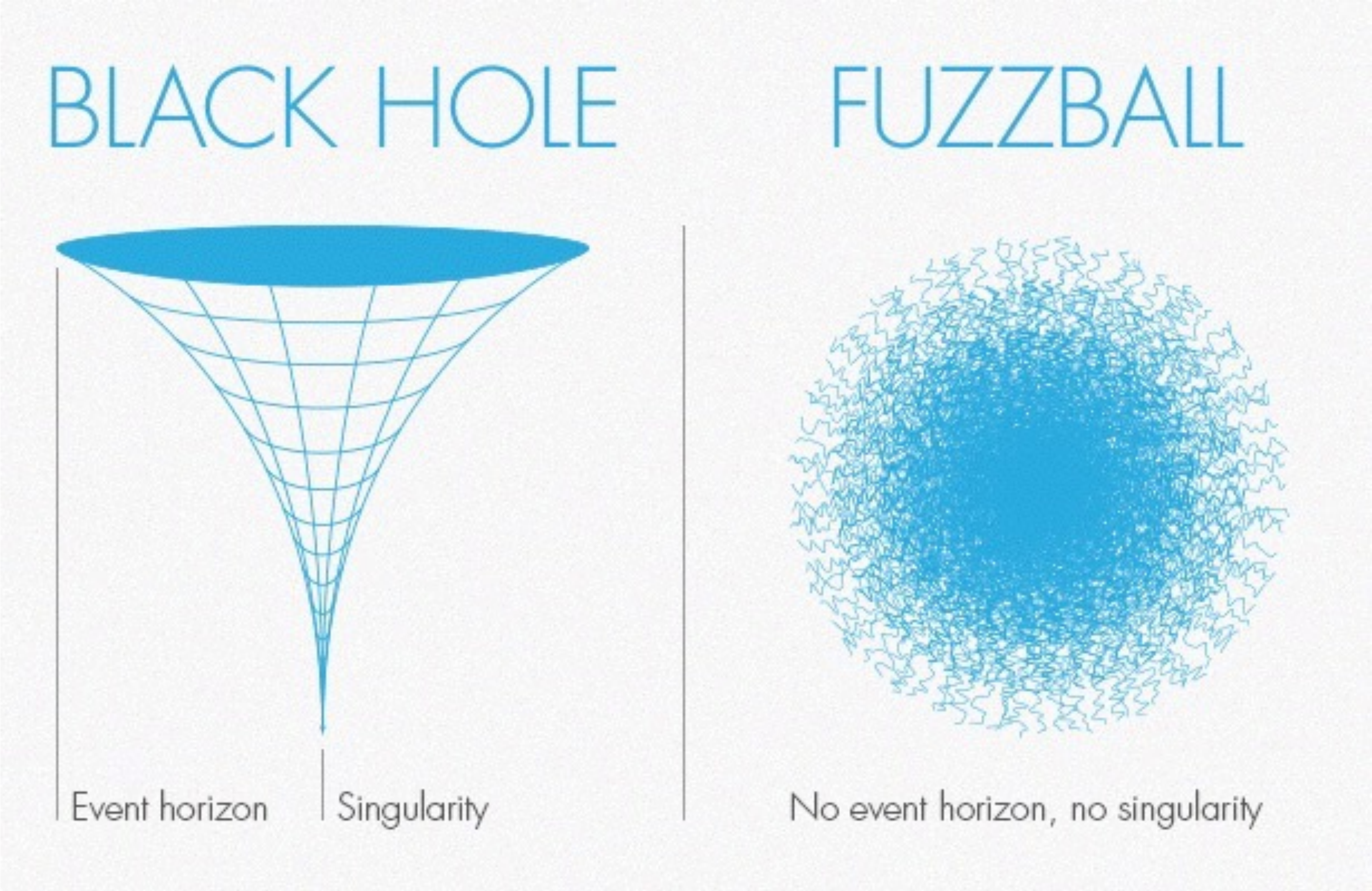
Geroch Group Matrices

$$\mathcal{M}(w) = \mathbf{1} + \sum_{i=1}^{N} \frac{A_i}{w - w_i}$$

- Simple poles; residues A_i are rank-1 matrices
- such that the matrix $\mathcal{M}(w)$ is in the affine group

Riemann Hilbert factorisation

- A Riemann-Hilbert factorisation allows us to construct the metric/spacetime fields from a Geroch group matrix
- Several examples in several theories have been worked out
- Asymptotic flatness in 4d as well as 5d are well under control



Vision

Future and comments

Structure at the horizon scale

- On increasing Newton's constant, horizon area grows, whereas microstructure shrinks. Microstates are planck size deep inside the black hole
- In string theory D-branes with shape modes gives structure that grows exactly in the same way as the size of a black hole
- Includes the famous Strominger-Vafa system, where string theory has had most success

Bena-Warner Microstate geometry program

- Systematic exploration of mechanisms that give smooth horizon-free solutions with horizon scale structure

- Over the last 10 years several developments (concisely reviewed in Gibbons-Warner 2014)

    1. importance of topology (bubbles and fluxes)
    2. Chern-Simons terms
    3. classical description of CFT microstates

- Current question: To what extent can bubbling geometries encode microstate structure.

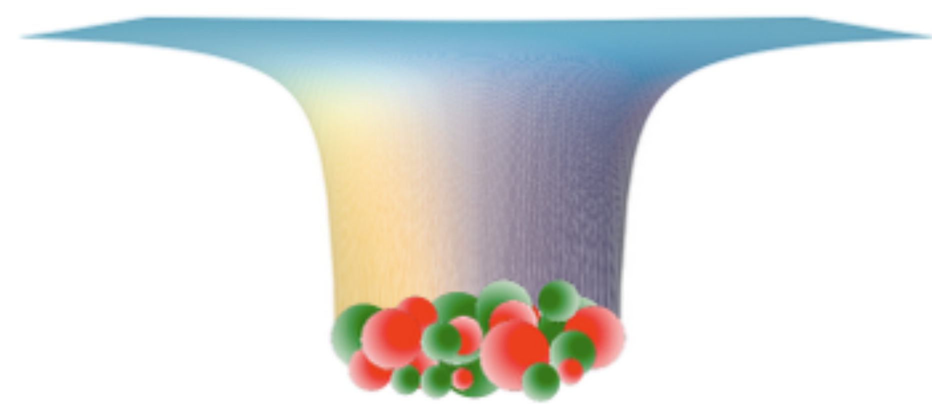
Supersymmetric microstates geometries

- Supersymmetric settings are computationally simpler to handle

- There are vast families of smooth, horizonless BPS, microstates geometries

- The geometry caps at the bottom of the throat. There is non-trivial topology. Bubbles arise at the original horizon scale

- Scaling solutions, with long AdS throats, these geometries come closest to the black hole picture



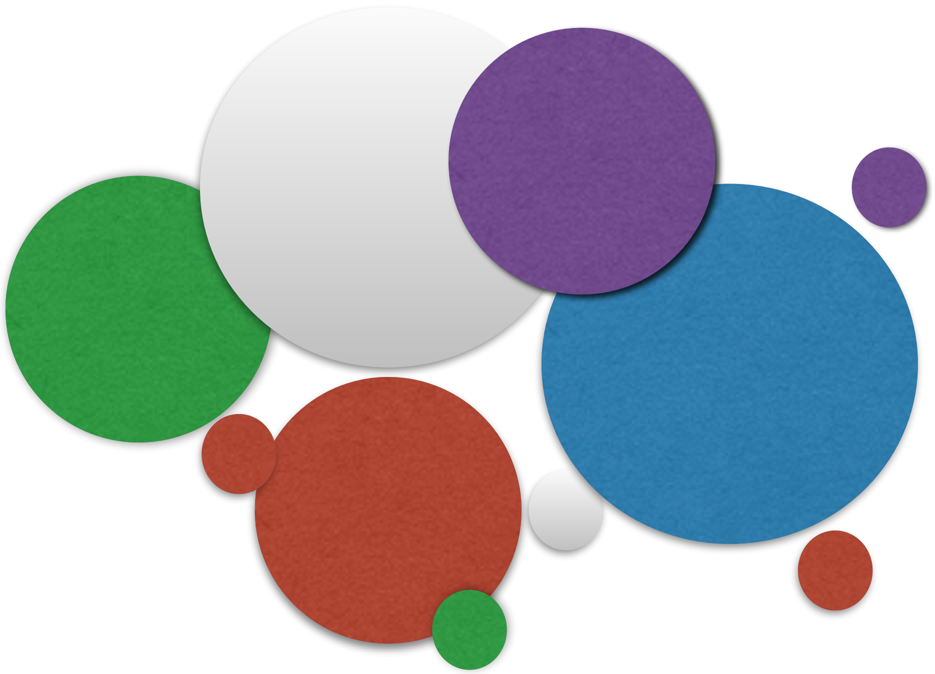
bubbled geometry

↕

CFT description

Non-supersymmetric Microstates

- The presently known non-extremal microstates are very far from being typical
- The logical way to approach seems to be first to construct multi-center solutions and then look at the scaling limit
- Technology to address these problems is sufficiently developed; some preliminary progress has also been made.



Thanks for you attention