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Problem of motion in GR & post-Newtonian theory

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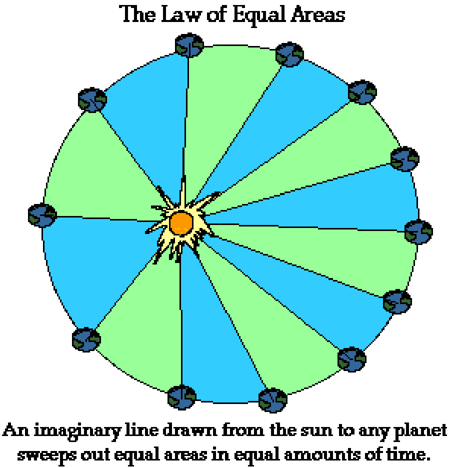
Kepler's three laws [1609 & 1619]

1. Law of Ellipses
2. Law of Equal Areas

$$\boxed{\ell = u - e \sin u \quad \text{(Kepler's equation)}}$$

3. Law of Harmonies

$$\boxed{\frac{P^2}{a^3} = \text{constant}}$$



Newton's universal law of gravitation [1687]

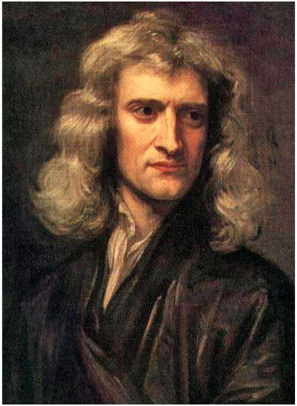
- Newton's gravitational law

$$\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A} \frac{G m_B}{r_{AB}^2}\,\boldsymbol{n}_{AB}$$

- Poisson's equation

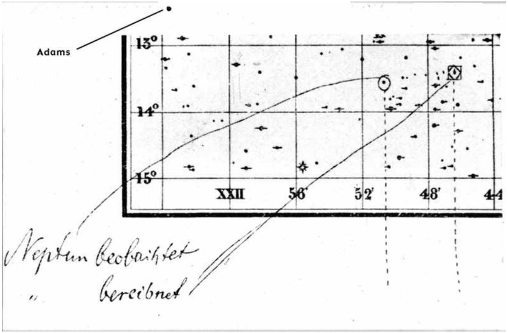
$$\Delta U = -4\pi\, G\,\rho$$

- Lagrangian and Hamiltonian formalisms
- Celestial mechanics and perturbation theory
- Non integrability of the three body problem
- Ergodicity and theory of chaotic systems
- Problem of the stability of the Solar System
- ...

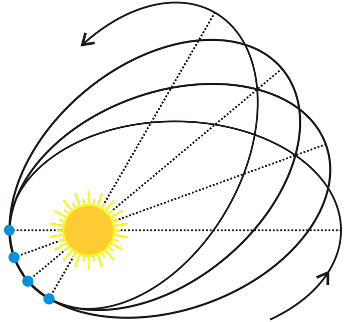


The triumph of Newtonian mechanics

Le Verrier [1846] predicts the position of the planet Neptune

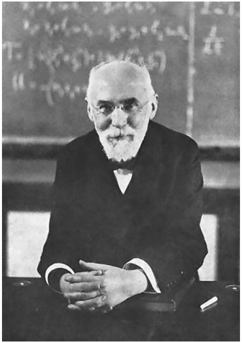


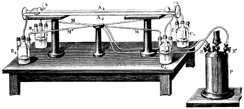
Mercury's periastron precession [Le Verrier 1859]



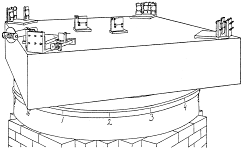
- Planet Vulcain located between Sun and Mercury?
- Presence of a ring of dust particles located in the ecliptic plane?
- Modification of Newton's law $1/r^2 \to 1/r^\alpha$?

Special relativity's revolution [1905]





[Fizeau 1851]



[Michelson & Morley 1887]

Einstein's equivalence principle [1911]

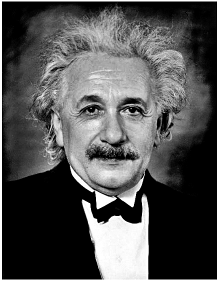
1. Weak equivalence principle. All test bodies have the same acceleration in a gravitational field, independently of their mass and internal structure
$$\boxed{m_i = m_g}$$
2. Local Lorentz invariance. The result of any non gravitational experiment performed in a freely falling frame is independent of the velocity of the frame
3. Local position invariance. The result of any non gravitational experiment in a freely falling frame is independent of the position in space and time

EEP is equivalent to a universal coupling of matter to the metric [Will 1993]

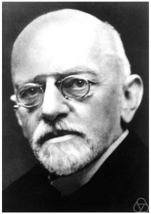
$$g_{\mu\nu}$$

which reduces to the Minkowski metric $\eta_{\alpha\beta}$ in freely falling frames

GR: the perfect theory [Einstein, November 1915]



[Einstein & Grossmann 1912]

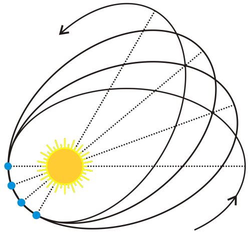


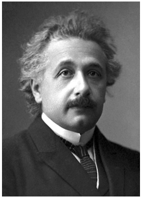
[Hilbert 1915]

$$\boxed{S_{\mathsf{GR}} = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\mathsf{m}}[g_{\mu\nu},\Psi]}_{\text{matter fields}}}$$

- Field equations imply by Bianchi's identity the matter equation of motion
- General covariance implies at linear order the gauge invariance of a massless spin-2 field (the "graviton")
- System of equations is a well-posed problem ("problème bien posé") in the sense of Hadamard [Choquet-Bruhat 1952]

Mercury's abnormal precession explained [1915]



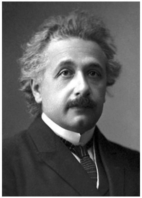


1. First relativistic corrections to Newtonian gravity imply

$$\boxed{\begin{aligned}\Delta\omega &= \frac{6\pi G\,M_\odot}{c^2\,a(1-e^2)}\left(\frac{2+2\gamma-\beta}{3}\right) + 3\pi J_2\left(\frac{R_\odot}{a(1-e^2)}\right)^2 \\ &= 43''/\text{century}\left[\frac{2+2\gamma-\beta}{3} + 2\,10^{-4}\left(\frac{J_2}{10^{-7}}\right)\right]\end{aligned}}$$

2. PPN parameters ($\gamma = \beta = 1$ in GR) [Eddington 1922, Nordtvedt 1968, Will 1972]
   - γ measures the spatial curvature
   - β measures the amount of non-linearity

Einstein's quadrupole formula [1916]



1. The GW amplitude is given by the first quadrupole formula

$$h_{ij}(\mathbf{x},t) = \frac{2G}{c^4 r}\left[\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t-\frac{r}{c}\right)\right]^{\mathrm{TT}} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

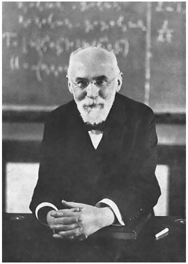
2. The total GW energy flux is given by the Einstein quadrupole formula

$$\boxed{\left(\frac{dE}{dt}\right)^{\mathrm{GW}} = \frac{G}{5c^5}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}}$$

3. The radiation reaction force is given by the third quadrupole formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1971]

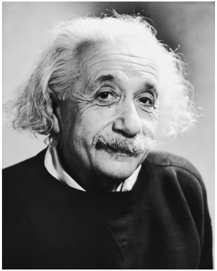
$$\boxed{\mathcal{F}_i^{\mathrm{RR}} = \frac{2G}{5c^5}\,\rho\, x^j\,\frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5}}$$

Post-Newtonian equations of motion [Lorentz & Droste 1917]



- Obtain the equations of motion of N bodies at the 1PN $\sim (v/c)^2$ order
- Even derive the 1PN Lagrangian!

Einstein-Infeld-Hoffmann equations [1938]



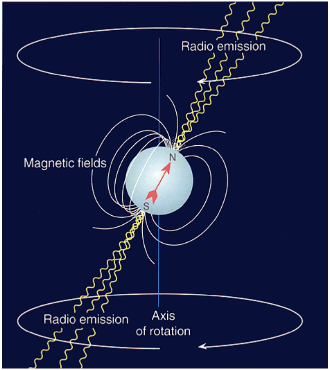
$$\boxed{\begin{aligned}\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\Bigg[1 - 4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}} - \sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1-\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right)\\ + \frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\Bigg]\\ + \sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}\end{aligned}}$$

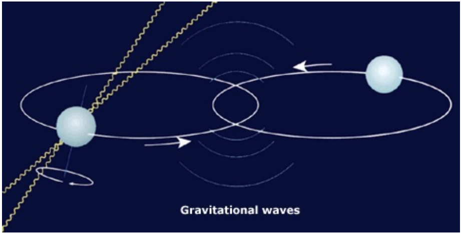
Post-Newtonian theory developing [1930s to 1970s]

[Fock 1939; Papapetrou 1951; Chandrasekhar 1965; Ehlers 1976; Thorne 1975; Will 1972; Damour 1980]

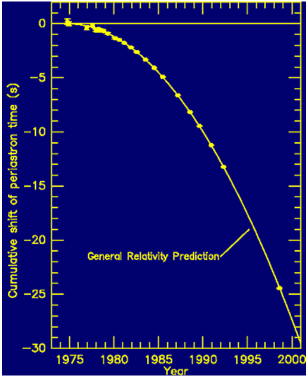
The binary pulsar PSR 1913+16


[Hulse & Taylor 1974]





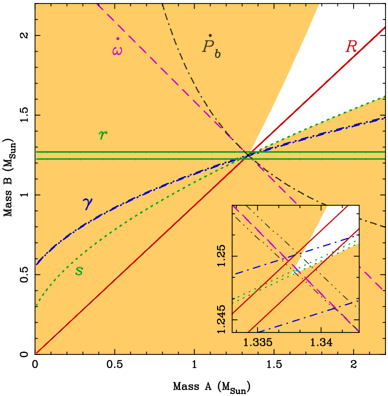
Orbital decay of the binary pulsar [Taylor & Weisberg 1982]



$$\boxed{\dot{P} = -\frac{192\pi}{5c^5}\,\frac{\mu}{M}\left(\frac{2\pi G\,M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}}$$

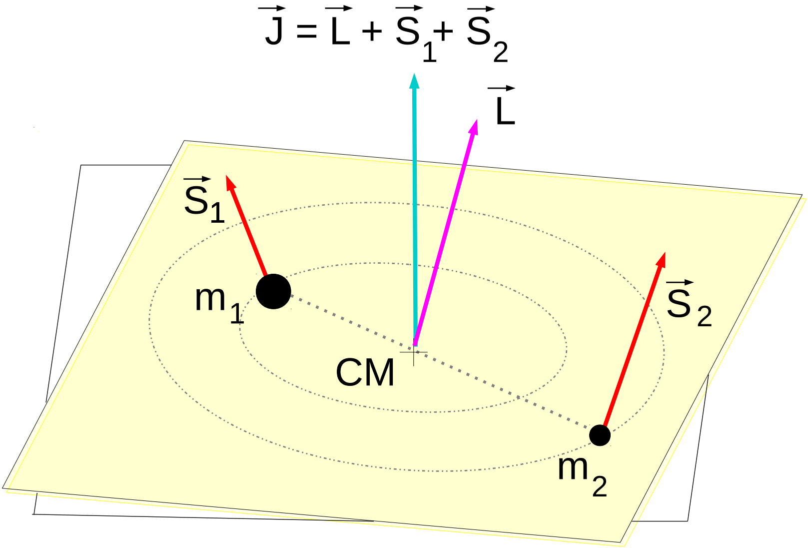
[Peters & Mathews 1963, Esposito & Harrison 1975; Wagoner 1975; Damour & Deruelle 1983]

30 years of GR effects in binary pulsars [e.g. Stairs 2003]

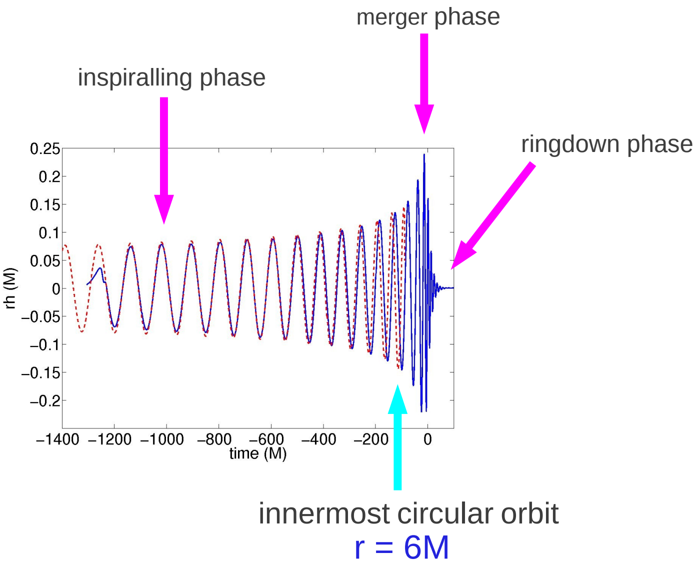


1. $\dot{\omega}$ relativistic advance of periastron
2. γ gravitational red-shift and second-order Doppler effect
3. r and s range and shape of the Shapiro time delay
4. $\dot{P}$ secular decrease of orbital period

Modelling the compact binary inspiral



The gravitational chirp of compact binaries



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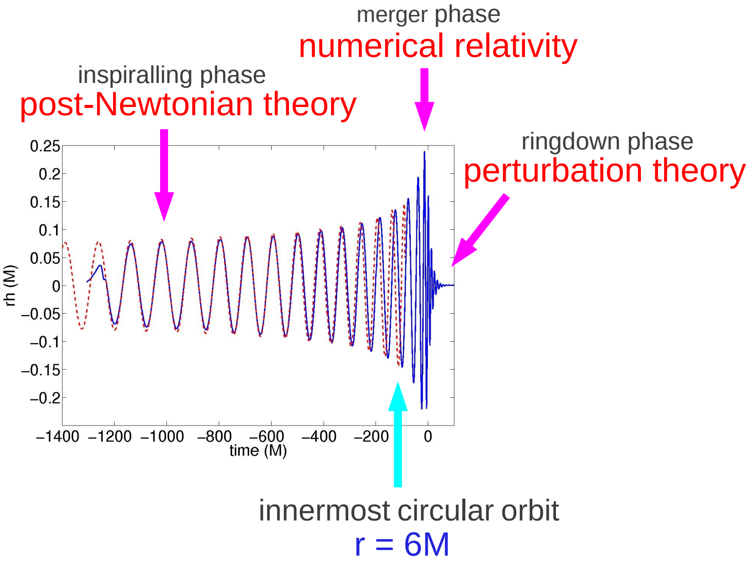
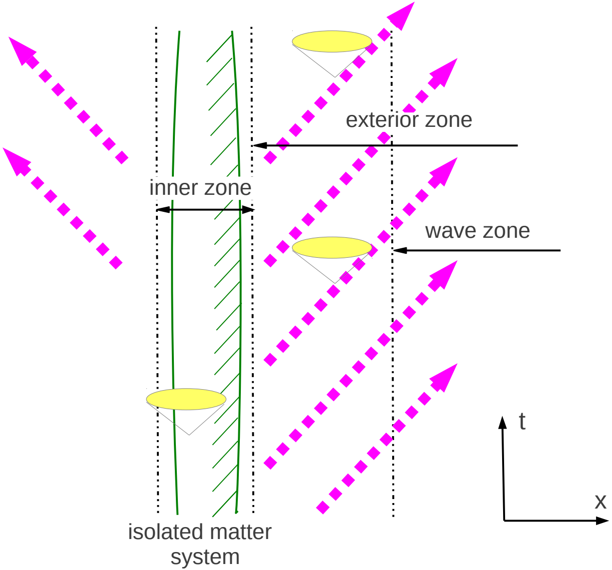
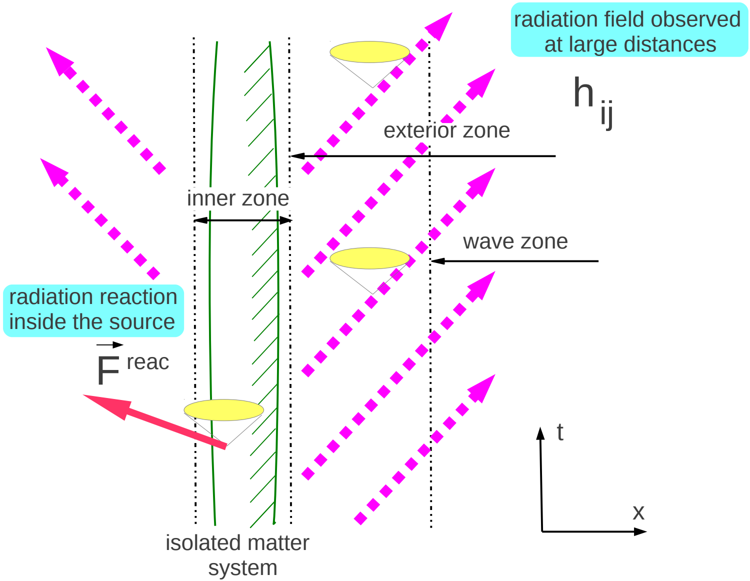


Isolated matter system in general relativity



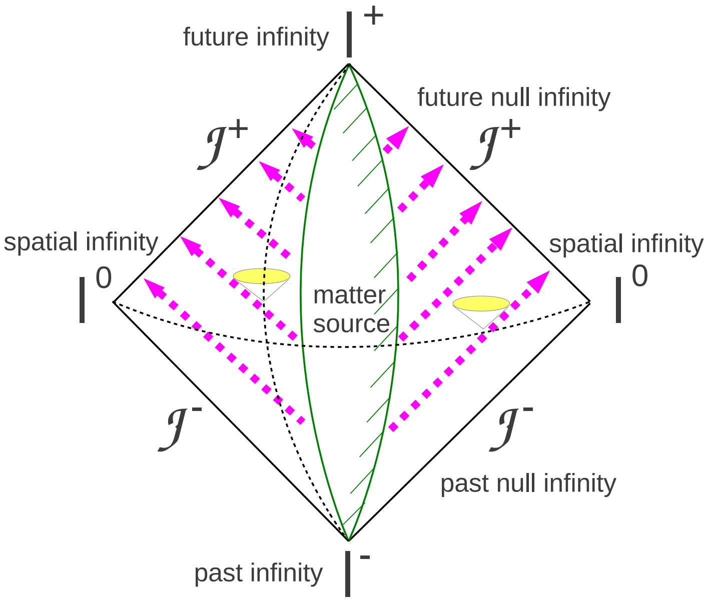
Isolated matter system in general relativity



Isolated matter system in general relativity

1. Generation problem
   - What is the gravitational radiation field generated in a detector at large distances from the source?
2. Propagation problem
   - Solve the propagation effects of gravitational waves from the source to the detector, including non-linear effects
3. Motion problem
   - Obtain the equations of motion of the matter source including all conservative non-linear effects
4. Reaction problem
   - Obtain the dissipative radiation reaction forces inside the source in reaction to the emission of gravitational waves

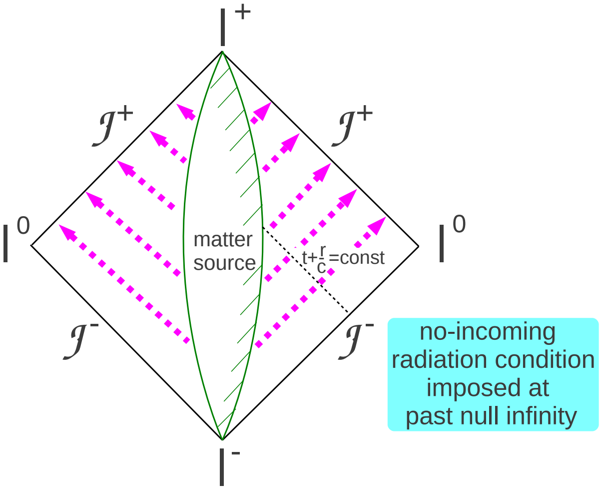
Conformal picture [Penrose 1963]



Asymptotic structure of space-time

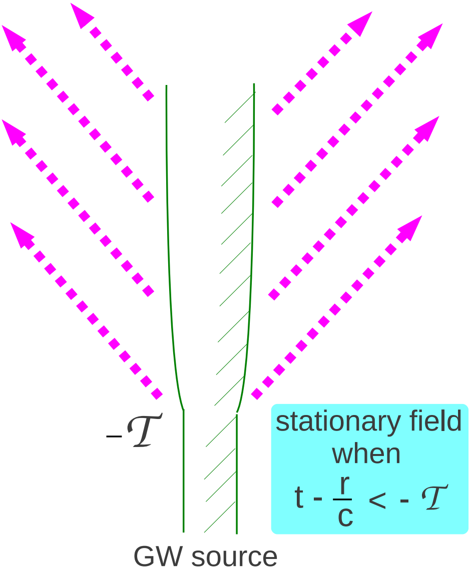
1. What is the struture of space-time far away from an isolated matter system?
2. Does a general radiating space-time satisfy rigourous definitions [Penrose 1963, 1965] of asymptotic flatness in general relativity?
3. How to relate the asymptotic structure of space-time [Bondi *et al.* 1962, Sachs 1962] to the matter variable and dynamics of an actual source?
4. How to impose rigourous boundary conditions on the edge of space-time appropriate to an isolated system?

No-incoming radiation condition



$$\boxed{\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}}\left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0}$$

Hypothesis of stationarity in the remote past



In practice all GW sources observed in astronomy (*e.g.* a compact binary system) will have been formed and started to emit GWs only from a finite instant in the past $-\mathcal{T}$

Post-Minkowskian expansion

[Bertotti & Plebanski 1960; Thorne & Kovàcs 1975; Damour & Esposito-Farèse 1997]

For a weakly self-gravitating isolated matter source

$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \qquad \begin{cases} M \text{ mass of source} \\ a \text{ size of source} \end{cases}$$

$$\sqrt{-g}g^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}}_{G \text{ labels the PM expansion}}$$

$$\boxed{\begin{aligned} &\Box h^{\alpha\beta}_{(n)} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta}_{(n)} + \overbrace{\Lambda^{\alpha\beta}_{(n)}[h_{(1)}, \cdots, h_{(n-1)}]}^{\text{know from previous iterations}} \\ &\partial_\mu h^{\alpha\mu}_{(n)} = 0 \end{aligned}}$$

Multipolar-post-Minkowskian expansion

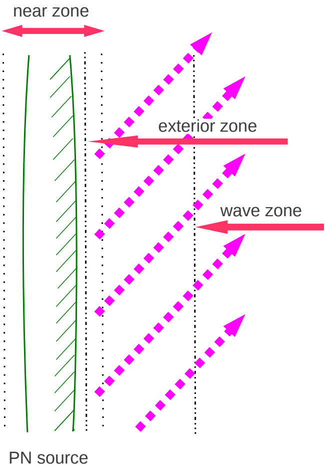
[Blanchet & Damour 1986; Blanchet 1987; Damour & Iyer 1991]

- Starts with the solution of the linearized equations outside an isolated source in the form of multipole expansions [Thorne 1980]
- An explicit MPM algorithm is constructed out of it by induction at any order n in the post-Minkowskian expansion
- A finite-part (FP) regularization based on analytic continuation is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

1. The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system
2. It is asymptotically simple in the sense of [Penrose 1963, 1965] and recovers the know asymptotic structure of radiative space-times at future null infinity [Bondi *et al.* 1962, Sachs 1962]

The MPM-PN formalism [Blanchet, Damour & Iyer 1995]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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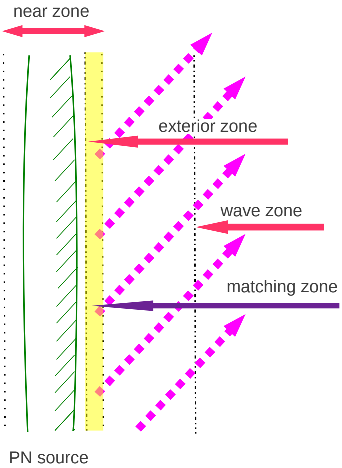


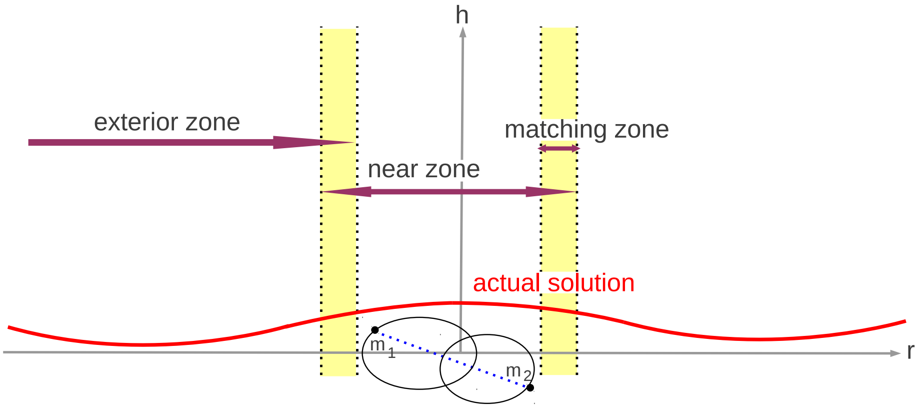
The matching equation

- This is a variant of the theory of matched asymptotic expansions
  [Lagerström *et al.* 1967; Kates 1980; Anderson *et al.* 1982]

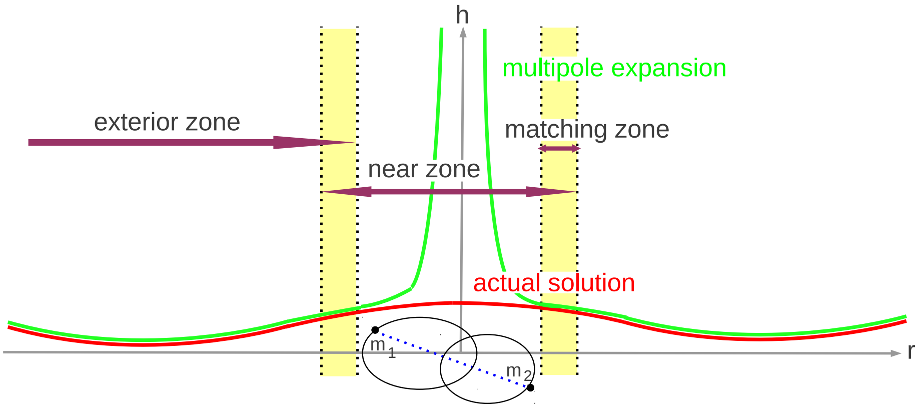
$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

  - Left side is the NZ expansion ($r \to 0$) of the exterior MPM field
  - Right side is the FZ expansion ($r \to \infty$) of the inner PN field
- The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field
- It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source

The matching equation



The matching equation



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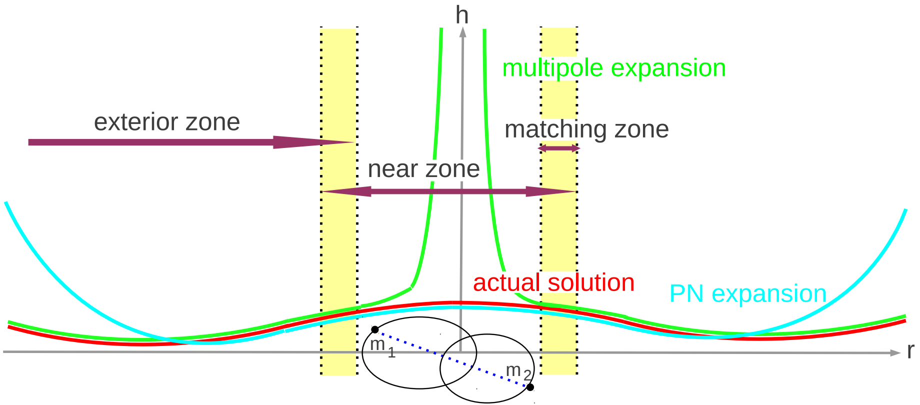
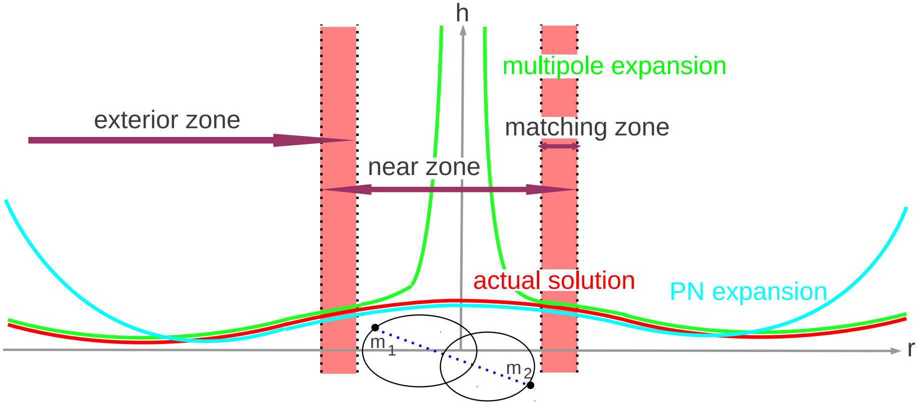


The matching equation



General solution for the multipolar field

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}\Box^{-1}_{\mathrm{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where $$M_L^{\mu\nu}(t) = \mathrm{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\,\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x}, t - zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem
- This is a formal PN solution *i.e.* a set of rules for generating the PN series regardless of the exact mathematocal nature of this series

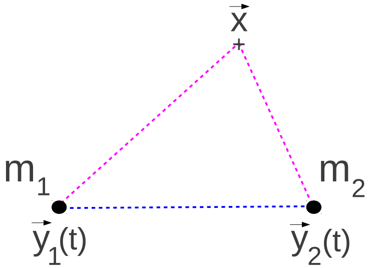
General solution for the inner PN field

$$\bar{h}^{\mu\nu} = \text{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r}\right\}}_{\text{homogeneous antisymmetric solution}}$$

where $$R_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_1^\infty \mathrm{d}z\,\gamma_\ell(z)\,\underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Problem of point particles



$$\boxed{U(\mathbf{x},t) = \frac{Gm_1}{|\mathbf{x}-\mathbf{y}_1(t)|} + \frac{Gm_2}{|\mathbf{x}-\mathbf{y}_2(t)|}}$$

$$\boxed{\frac{\mathrm{d}^2\mathbf{y}_1}{\mathrm{d}t^2} = (\boldsymbol{\nabla} U)\,(\mathbf{y}_1(t),t) \stackrel{?}{=} -Gm_2\frac{\mathbf{y}_1-\mathbf{y}_2}{|\mathbf{y}_1-\mathbf{y}_2|^3}}$$

- For extended bodies the self-acceleration of the body cancels out by Newton's action-reaction law
- For point particles one needs a self-field regularization to remove the infinite self-field of the particle

Dimensional regularization [t'Hooft & Veltman 1972; Bollini & Giambiagi 1972]

- Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

- For two point-particles $\rho = m_1 \delta(\mathbf{x}-\mathbf{y}_1) + m_2 \delta(\mathbf{x}-\mathbf{y}_2)$ where δ is the d-dimensional Dirac function we get

$$\boxed{U(\mathbf{x},t) = \frac{2(d-2)k}{d-1} \left(\frac{G m_1}{|\mathbf{x}-\mathbf{y}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x}-\mathbf{y}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}}$$

- Computations are performed when $\Re(d)$ is a large negative complex number so as to kill all self-terms, and the result is analytically continued for any $d \in \mathbb{C}$ except for poles occuring at integer values of d

3.5PN energy flux of compact binaries

[Blanchet, Faye, Iyer & Joguet 2002; Arun *et al.* 2004]

$$\boxed{\begin{aligned}\mathcal{F}^{\mathrm{GW}} = -\frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} \\ & + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \underbrace{\left[\cdots\right] x^{5/2}}_{\text{2.5PN tail}} \\ & + \underbrace{\left[\cdots\right] x^3}_{\substack{\text{3PN} \\ \text{includes a tail-of-tail}}} + \underbrace{\left[\cdots\right] x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(x^4\right) \Bigg\}\end{aligned}}$$

The orbital frequency and phase for quasi-circular orbits are deduced from an energy balance argument

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathrm{GW}}$$

3.5PN dominant gravitational wave modes

[Faye, Marsat, Blanchet & Iyer 2012; Faye, Blanchet & Iyer 2014]

$$
\boxed{\begin{aligned}
h_{22} &= \frac{2G\,m\,\nu\,x}{R\,c^2}\sqrt{\frac{16\pi}{5}}\,e^{-2\mathrm{i}\psi}\Bigg\{1 + x\left(-\frac{107}{42}+\frac{55\nu}{42}\right) + 2\pi x^{3/2} \\
&\qquad + x^2\left(-\frac{2173}{1512}-\frac{1069\nu}{216}+\frac{2047\nu^2}{1512}\right) \\
&\qquad + \underbrace{[\cdots]\,x^{5/2}}_{\text{2.5PN}} + \underbrace{[\cdots]\,x^{3}}_{\text{3PN}} + \underbrace{[\cdots]\,x^{7/2}}_{\text{3.5PN}} + \mathcal{O}\left(x^4\right)\Bigg\} \\
h_{33} &= \cdots \\
h_{31} &= \cdots
\end{aligned}}
$$

Tail contributions in this expression are factorized out in the phase variable

$$\psi = \phi - \frac{2GM\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)$$

4PN spin-orbit effects in the orbital frequency

[Marsat, Bohé, Faye, Blanchet & Buonanno 2013]

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\,\nu\, x^{5/2}\Bigg\{\overbrace{1 + x\,[\cdots] + x^{3/2}\,[\cdots] + x^2\,[\cdots] + x^{5/2}\,[\cdots] + x^3\,[\cdots]}^{\text{non-spin terms}}$$
$$+ \underbrace{[\cdots]\,x^{3/2}}_{\text{1.5PN SO}} + \underbrace{[\cdots]\,x^2}_{\text{2PN SS}} + \underbrace{[\cdots]\,x^{5/2}}_{\text{2.5PN SO}} + \underbrace{[\cdots]\,x^3}_{\text{3PN SO}_{\text{tail}}\text{ \& SS}}$$
$$+ \underbrace{[\cdots]\,x^{7/2}}_{\text{3.5PN SO}} + \underbrace{[\cdots]\,x^4}_{\text{4PN S0}_{\text{tail}}\text{ \& SS}} + \mathcal{O}\left(x^4\right)\Bigg\}$$

Leading SO and SS terms due to [Kidder, Will & Wiseman 1993; Kidder 1995]

Summary of current PN results

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN) [BDI, Faye, Arun]	4PN non-spin 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (L) SSS	3.5PN non-spin 4PN (NNL) SO 3PN (NL) SS 3.5PN (L) SSS	3PN non-spin 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian [Jaranowski, Schäfer, Damour] [Steinhoff, Hergt, Hartung]	4PN non-spin 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (L) SSS		
Effective Field Theory (EFT) [Goldberger, Rothstein] [Porto, Foffa, Sturani, Ross, Levi]	3PN non-spin 2.5PN (NL) SO 4PN (NNL) SS	2PN non-spin 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE) [Will, Wiseman, Kidder, Pati]	2.5PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS
Surface Integral [Itoh, Futamase, Asada]	3PN non-spin		

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THE 4PN EQUATIONS OF MOTION

Based on a collaboration with

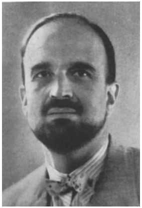
Laura Bernard, Alejandro Bohé, Guillaume Faye & Sylvain Marsat

4PN equations of motion of compact binaries

$$\boxed{\begin{aligned}\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = &-\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12}\\ &+\overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]\boldsymbol{n}_{12}+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}}\\ &+\underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}}+\underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}}+\underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}}+\underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\\text{radiation reaction}}}+\underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\\text{conservative \& radiation tail}}}+\mathcal{O}\left(\frac{1}{c^9}\right)\end{aligned}}$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001]	ADM Hamiltonian
	[Blanchet & Faye 2000; de Andrade, Blanchet & Faye 2001]	Harmonic equations of motion
	[Itoh, Futamase & Asada 2001; Itoh & Futamase 2003]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye & Marsat 2015]	Fokker Lagrangian

Fokker action of N particles [Fokker 1929]

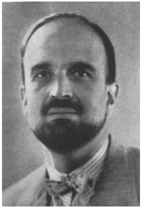


Start with the gauge-fixed action for a system of N point particles

$$S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[\overbrace{g^{\mu\nu}\left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho}-\Gamma^{\rho}_{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right)}^{\text{Landau-Lifchitz form of the action}}\underbrace{-\frac{1}{2}g_{\mu\nu}\Gamma^{\mu}\Gamma^{\nu}}_{\text{Gauge-fixing term}}\Big] + S_m$$

$$S_m = -\sum_A \underbrace{m_A c^2 \int \mathrm{d}t\,\sqrt{-(g_{\mu\nu})_A\, v_A^{\mu} v_A^{\nu}/c^2}}_{N\text{ point particles without spin}}$$

Fokker action of N particles [Fokker 1929]



The Fokker action is obtained by inserting an explicit PN solution of the relaxed Einstein field equations

$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots)$$

$$\boxed{\begin{aligned} S_{\mathsf{F}}\left[\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots\right] &= \int \mathrm{d}^4x\,\overline{\mathcal{L}}_g\left[\mathbf{x};\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots\right] \\ &\quad - \sum_A m_A c^2 \int \mathrm{d}t \sqrt{-\overline{g}_{\mu\nu}\left(\boldsymbol{y}_A(t);\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots\right)\,v_A^\mu v_A^\nu/c^2} \end{aligned}}$$

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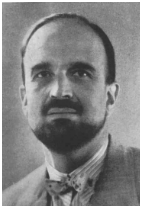

The PN equations of motion of the N particles in the field generated by the particles themselves (self-gravitating system) are obtained as

$$\boxed{\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{y}_A} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{y}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

Fokker action in the PN approximation

- The Fokker action is split into a PN (near-zone) term plus a contribution involving the multipole (far-zone) expansion

$$S^g_{\mathsf{F}} = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g + \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)$$

- The multipole term gives zero for any "instantaneous" term

$$\int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)\Big|_{\text{inst}} = 0$$

thus only "hereditary" terms contribute and they are at least 5.5PN

- Finally we obtain

$$\boxed{S^g_{\mathsf{F}} = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^4x \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g}$$

where the constant r_0 represents an IR cut-off scale and plays a crucial role at the 4PN order

Gravitational wave tail effect at the 4PN order

- At 4PN order there is an imprint of gravitational wave tails in the local (near-zone) dynamics of the source
- This leads to a non-local-in-time contribution in the Fokker action

$$\boxed{S_{\mathsf{F}}^{\mathsf{tail}} = \frac{G^2 M}{5c^8} \underset{s_0}{\mathsf{Pf}} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')}$$

- The constant s_0 is *a priori* different from the IR scale r_0 but posing

$$s_0 = r_0\, e^{-\alpha}$$

  we find that r_0 finally cancels out so the result is IR finite
- The remaining parameter α is a pure number and turns out to be an "ambiguity" in the present formalism

The method "$n+2$"

- Adopt as basic gravitational variables

$$\overline{h} \equiv \left(\overline{h}^{00} + \overline{h}^{ii}, \overline{h}^{0i}, \overline{h}^{ij}\right)$$

- Suppose that $\overline{h}$ is known up to order $1/c^{n+2}$ thus

$$\boxed{\overline{h} = \overline{h}_n + \delta\overline{h}_n \quad \text{where} \quad \delta\overline{h}_n = \mathcal{O}\left(\frac{1}{c^{n+3}}\right)}$$

- Expand the Fokker action around the known solution

$$S_\mathsf{F}[\overline{h}] = S_\mathsf{F}[\overline{h}_n] + \underbrace{\int \mathrm{d}^4x \, \frac{\delta S_\mathsf{F}}{\delta \overline{h}}[\overline{h}_n]\,\delta\overline{h}_n + \mathcal{O}(\delta\overline{h}_n^2)}_{\text{is at least of order } \mathcal{O}(1/c^{2n+2})}$$

- Thus the Fokker action is known up to $1/c^{2n}$ *i.e.* nPN order

Conserved energy for circular orbits at 4PN order

- The energy for circular orbits at the 4PN order in the small mass ratio limit is known from self-force calculations of the redshift variable [Detweiler 2008]
- This permits to fix the ambiguity parameter α and to complete the 4PN equations of motion
- For instance the 4PN invariant energy for circular orbits reads

$$\boxed{\begin{aligned}E^{\text{4PN}} = -\frac{\mu c^2 x}{2}\Bigg\{&1+\left(-\frac{3}{4}-\frac{\nu}{12}\right)x+\left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{\nu^2}{24}\right)x^2\\&+\left(-\frac{675}{64}+\left[\frac{34445}{576}-\frac{205}{96}\pi^2\right]\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3\\&+\left(-\frac{3969}{128}+\left[-\frac{123671}{5760}+\frac{9037}{1536}\pi^2+\frac{896}{15}\gamma_{\text{E}}+\frac{448}{15}\ln(16x)\right]\nu\right.\\&\qquad\left.+\left[-\frac{498449}{3456}+\frac{3157}{576}\pi^2\right]\nu^2+\frac{301}{1728}\nu^3+\frac{77}{31104}\nu^4\right)x^4\Bigg\}\end{aligned}}$$

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