BLACK HOLE ENTROPY

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Abstract

We review black hole entropy with special reference to the old euclidean quantum gravity, the intermediate brick wall approach and the modern loop quantum gravity.

1. Introduction

In Einstein's theory of gravitation, the gravitational field due to a point mass is described by a metric which has many interesting properties. Its black hole features have been known for a very long time, but in the seventies it began to appear that thermodynamic concepts like temperature and entropy were also associated with it. Gradually it was realized that these were quantum effects. But the degrees of freedom associated with the entropy could not be easily identified. Many suggestions have been made: we shall discuss the entanglement entropy approach and the more recent loop quantum gravity.

After summarizing black hole mechanics, we consider the euclidean quantum gravity approach for both non-extremal and extremal black holes. There are indications of discontinuity between the two kinds, which arises in one way of quantization of the classical theory. An alternative way leads to the Bekenstein-Hawking formula even for extremal black holes.

1. Introduction

In order to understand the origin of black hole entropy, the entropy of fields in black hole backgrounds was studied. This is identified as entanglement entropy and arises because the region in the interior of the horizon has to be traced over.

More recently, attempts have been made to formulate a quantum theory of gravity itself. Black hole entropy has also been calculated in this loop quantum gravity approach. This will be discussed in detail.

1. Introduction

2. Euclidean quantum gravity

2.1. Preliminaries

Precursor of idea of entropy of black holes – **area theorem**: area of the horizon of a system of black holes always increases in a class of spacetimes

more generally, *laws of black hole mechanics* analogous to laws of thermodynamics

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- zeroeth law: surface gravity κ remains constant on horizon of black hole
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  $A =$ area of horizon
  $\phi =$ potential at horizon of black hole (mass M, charge Q)

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- first law: $\frac{\kappa dA}{8\pi} = dM - \phi dQ$,
  $A =$ area of horizon
  $\phi =$ potential at horizon of black hole (mass M, charge Q)
- second law: area of horizon of system always increases in spacetimes which are predictable from partial Cauchy hypersurfaces..

For charged black hole,

$$r_\pm = M \pm \sqrt{M^2 - Q^2},$$

$$\kappa = \frac{r_+ - r_-}{2r_+^2},\ \phi = Q/r_+,\ A = 4\pi r_+^2.$$

When these observations were made, no obvious connection with thermodynamics – only a matter of analogy

But the existence of a horizon imposes limitation on amount of information available and hence may lead to an entropy, which should then be measured by geometric quantity associated with horizon, namely area

$\Rightarrow$ **upto a factor**, $A \propto$ entropy, $\frac{\kappa}{8\pi} \propto$ temperature

Interpretation not fully convincing – but quantum theory found to cause dramatic changes in black hole spacetimes: scalar field theory in Schwarzschild black hole background indicates radiation of particles at temperature

$$T = \frac{\hbar}{8\pi M} = \frac{\hbar\kappa}{2\pi}.$$

$\Rightarrow$ connection of laws of black hole mechanics with thermodynamics, fixes scale factor: involves Planck's constant: quantum effect

For Schwarzschild black hole, first law of *thermodynamics* simplifies:

$$TdS = dM.$$

can be integrated:

$$S = \frac{4\pi M^2}{\hbar} = \frac{A}{4\hbar}.$$

$T=\frac{\hbar\kappa}{2\pi}$ generally valid for black holes having $g_{tt}\sim(1-\frac{r_h}{r})$
first law becomes

$$Td\frac{A}{4\hbar}=dM-\phi dQ.$$

Comparison with first law of *thermodynamics*

$$TdS=dM-\phi dQ$$

leads to identification

$$S=\frac{A}{4\hbar}.$$

2.2. Non-extremal black holes

Grand partition function for euclidean charged black holes:

$$Z_{\rm grand}\equiv e^{-\frac{M-TS-\phi Q}{T}}\approx e^{-I/\hbar},$$

functional integral over all configurations (consistent with appropriate boundary conditions) semiclassically approximated by integrand;

classical action I can be calculated:

for euclidean Reissner - Nordström black hole in manifold $\mathcal{M}$ with boundary subsequently taken to infinity,

$$\begin{aligned} I &= -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}R + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{\gamma}(K-K_0) + \\ &\quad \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}. \end{aligned}$$

γ: induced metric on boundary $\partial\mathcal{M}$, K extrinsic curvature, from which a subtraction has to be made

First term of action vanishes (Einstein's equations $\Rightarrow R = 0$)

To evaluate second term, take boundary of manifold at $r = r_B \to \infty$

$$\begin{aligned} K &= -\frac{1}{\sqrt{g_{tt}}r^2}\frac{1}{\sqrt{g_{rr}}}\frac{d}{dr}(\sqrt{g_{tt}}r^2) \\ &= -\frac{1}{r^2}\frac{d}{dr}[(1-\frac{M}{r}+\cdots)r^2] \end{aligned}$$

$$= -\frac{1}{r^2}\frac{d}{dr}(r^2 - Mr),$$

$$\int d^3x\sqrt{\gamma} = \int dt(1 - \frac{M}{r} + \cdots)4\pi r^2.$$

$\int d^3x\sqrt{\gamma}K$ diverges as $r \to \infty$:
can be cured by taking K minus flat space contribution $K_0 = -\frac{1}{r^2}\frac{d}{dr}r^2$
second piece of action becomes

$$-\frac{1}{8\pi}\int dt(1 - \frac{M}{r} + \cdots)4\pi r^2\frac{1}{r^2}\frac{d}{dr}(-Mr)|_{r=r_B\to\infty}$$
$$= -\frac{1}{2}\int dt(-M) = \frac{1}{2}\beta M.$$

Euclidean time t has to go over one period $0 \to \beta = \frac{2\pi}{\kappa}$ to avoid *conical singularity* at horizon

Third term becomes

$$\begin{aligned} & -\frac{1}{16\pi}\int dt.4\pi\int drr^2.2.\frac{Q^2}{r^4} \\ = & -\frac{1}{2}\int dt\frac{Q^2}{r_+} \\ = & -\frac{1}{2}\beta Q\phi, \end{aligned}$$

where $\phi =$ electrostatic potential (negative sign in euclidean solution)

Finally,

$$I = \frac{1}{2}\beta(M - Q\phi) = \frac{A}{4}.$$

$$M = T(S + \frac{I}{\hbar}) + \phi Q = T(S + \frac{A}{4\hbar}) + \phi Q.$$

Smarr formula:

$$M = \frac{\kappa A}{4\pi} + \phi Q = T\frac{A}{2\hbar} + \phi Q$$

$\Rightarrow S = \frac{A}{4\hbar}$

2.3. Extremal black holes

Extremal: $r_+ = r_-, Q = M, \phi = 1$

Special interest: topology changes discontinuously in passage from (euclidean) non-extremal to extremal case

Action

$$I = \frac{1}{2}\beta(M - Q\phi) = 0,$$

$$M = T(S + \frac{I}{\hbar}) + \phi Q = TS + M \Rightarrow S = 0$$

β assumed finite;

note: $\lim_{Q\to M} \beta = \infty$

but no conical singularity in extremal case, so no reason to fix euclidean temperature, should be arbitrary

Here, quantum theory based exclusively on extremal topology

Alternative quantization: *sum over topologies*

Here temperature β^{-1} and chemical potential Φ specified as inputs at boundary of manifold r_B, and mass M and charge Q of black hole calculated as functional integral average.

Definition of extremality $Q = M$ imposed on these:

extremalization after quantization, as opposed to *quantization after extremalization.*

Spherically symmetric class of metrics considered;

boundary conditions:

$$g_{tt}(r_+) = 0,\ 2\pi\sqrt{g_{tt}(r_B)} = \beta.$$

$$A_t(r_+) = 0,\ A_t(r_B) = \frac{\beta\Phi}{2\pi i}.$$

(vector potential taken = zero)

Another boundary condition reflects extremal/non-extremal nature:

$$\frac{1}{\sqrt{g_{rr}(r_+)}}\frac{d}{dr_+}\sqrt{g_{tt}(r_+)} = 1\ldots\text{in non}-\text{extremal case},$$
$$\text{but} = 0\ldots\text{in extremal case}.$$

Variation of action with boundary conditions leads to reduced versions of Einstein - Maxwell equations, solution has mass parameter m and charge q arbitrary.

$$I = \beta(m - q\Phi) - \pi(m + \sqrt{m^2 - q^2})^2 \text{ for non}-\text{extremal bc},$$
$$I = \beta(m - q\Phi) \text{ for extremal bc}.$$

Partition function is of form

$$\sum_{\text{topologies}} \int d\mu(m) \int d\mu(q) e^{-I(q,m)},$$

with I appropriate for non-extremal/extremal q.

Semiclassical approximation involves replacing double integral by maximum value of integrand, *i.e.*, by $e^{-I_{min}}$, where I_{min} is classical action for *non-extremal* case, *minimized* with respect to $q,\ m$, yielding function of $\beta,\ \Phi$
$\Rightarrow S = A/4$ for all values of $\beta,\ \Phi$.
Averages $Q,\ M$, calculated from $\beta,\ \Phi$. Extremal limit reached for limiting values

$$\beta \to \infty, \ \ |\Phi| \to 1, \text{ with } \gamma \equiv \beta(1 - |\Phi|) = 2\pi M(\text{finite})$$

for the ensemble parameters $\beta,\ \Phi$. Classical action

$$I = \frac{\gamma^2}{4\pi} = \pi M^2.$$

$$Z \equiv e^{S-\gamma M/\hbar} = e^{-\pi M^2/\hbar},$$

continuing to correspond to $S = \frac{A}{4\hbar}$.

2. Euclidean quantum gravity

3. Matter in black hole background

To study entropy of scalar field in background provided by black hole, employ brick-wall boundary condition, wave function cut off just outside the horizon

$$\varphi(x) = 0 \qquad \text{at } r = r_h + \epsilon$$

ϵ= ultraviolet cut-off.
Also infrared cut-off (box):

$$\varphi(x) = 0 \qquad \text{at } r = L >> r_h$$

Static, spherically symmetric black hole spacetime

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)d\Omega^2$$

r- dependent radial wave number for particles with mass m, energy E and orbital angular momentum l:

$$k_r^2(r, l, E) = g_{rr}[-g^{tt}E^2 - l(l+1)g^{\theta\theta} - m^2] \geq 0$$

3. Matter in black hole background

semiclassical quantization condition

$$\frac{1}{\pi}\int_{r_h+\epsilon}^{L} dr\ k_r(r,l,E) = n_r \text{ integral}$$

Free energy F at inverse temperature β given by sum over single particle states

$$\begin{aligned}
\beta F &= \sum_{n_r,l,m_l} \log(1-e^{-\beta E}) \\
&\approx \int dl\ (2l+1)\int dn_r \log(1-e^{-\beta E}) \\
&= -\int dl\ (2l+1)\int d(\beta E)\ (e^{\beta E}-1)^{-1} n_r \\
&= -\frac{\beta}{\pi}\int dl\ (2l+1)\int dE\ (e^{\beta E}-1)^{-1}\int_{r_h+\epsilon}^{L} dr\ g_{rr}^{1/2} \\
&\quad \sqrt{-g^{tt}E^2 - l(l+1)g^{\theta\theta} - m^2}
\end{aligned}$$

3. Matter in black hole background

$$= \quad -\frac{2\beta}{3\pi}\int_{r_h+\epsilon}^{L} dr\ g_{rr}^{1/2} g_{\theta\theta}(-g_{tt})^{-3/2}$$

$$\int dE\ (e^{\beta E}-1)^{-1}[E^2+g_{tt}m^2]^{3/2}.$$

Limits of integration for l, E such that arguments of square roots are nonnegative.

l integration explicit, E integral to be approximated

Contribution to r integral from large r also present in flat spacetime

$$F_0 = -\frac{2}{9\pi}L^3\int_m^\infty dE\frac{(E^2-m^2)^{3/2}}{e^{\beta E}-1}$$

not relevant.

Contribution from small r singular in limit $\epsilon \to 0$.

3. Matter in black hole background

For non-extremal black hole, $g_{rr} \propto (r - r_h)^{-1}$, $g_{tt} \propto (r - r_h)$, $g_{\theta\theta}$ regular:

$$F_{sing} \approx -\frac{2\pi^3}{45\epsilon\beta^4}[(r - r_h)g_{rr}]^{1/2}(-\frac{g_{tt}}{r - r_h})^{-3/2}g_{\theta\theta}|_{r=r_h},$$

with corrections involving $m^2\beta^2$

Entropy

$$S = \beta^2\frac{\partial F}{\partial\beta}.$$

$$S_{sing} = \frac{8\pi^3}{45\beta^3\epsilon}[(r - r_h)g_{rr}]^{1/2}(-\frac{g_{tt}}{r - r_h})^{-3/2}g_{\theta\theta}|_{r=r_h}.$$

3. Matter in black hole background

Now Hawking temperature

$$\frac{1}{\beta} = \frac{1}{2\pi}(g_{rr})^{-1/2}\frac{\partial}{\partial r}(-g_{tt})^{1/2}|_{r=r_h}$$
$$= \frac{1}{4\pi}(g_{rr})^{-1/2}(-g_{tt})^{-1/2}\frac{\partial}{\partial r}(-g_{tt})|_{r=r_h}$$
$$= \frac{1}{4\pi}[(r-r_h)g_{rr}]^{-1/2}(-\frac{g_{tt}}{r-r_h})^{1/2}|_{r=r_h}.$$

and proper radial width ($d\tilde{r}^2 \equiv g_{rr}dr^2$):

$$\tilde{\epsilon} = \tilde{r}(r_h+\epsilon) - \tilde{r}(r_h) \approx 2\epsilon^{1/2}[(r-r_h)g_{rr}]^{1/2}|_{r=r_h}$$

$$\Rightarrow S_{sing} = \frac{1}{90\tilde{\epsilon}^2}g_{\theta\theta}|_{r=r_h} = \frac{1}{360\pi\tilde{\epsilon}^2}\text{Area}.$$

This area factor crucially depends on behaviour of metric near horizon, valid only for non-extremal black holes; does **not** emerge in extremal case.

3. Matter in black hole background

4. Loop quantum gravity
4.1. Preliminaries

Framework for calculation of black hole entropy: loop quantum gravity or quantum geometry approach.

A classical "isolated horizon": quantum states built up by associating spin variables with "punctures" on horizon.
Entropy obtained by counting all possible states consistent with given area (particular eigenvalue of the area operator).

Generic configuration has s_j punctures with spin $j, j = 1/2, 1, 3/2....$

$$2\sum_j s_j\sqrt{j(j+1)} = A,$$

$A =$ horizon area in special units:

$$4\pi\gamma\ell_P^2 = 1,$$

γ= Barbero-Immirzi parameter, ℓ_P= Planck length.

Spin projection constraint

$$\sum m = 0, \quad \text{all punctures}$$

$$m \in \{-j, -j+1, ...j\} \text{ for puncture with spin } j$$

4.2. Spin 1/2

For simplicity first consider spin 1/2 on *each* puncture.

Punctures have to be considered distinguishable.

Number of punctures n with spin 1/2 given by

$$2n\sqrt{\frac{3}{4}} = A,$$

so *if we neglect spin projection constraint*, entropy

$$n \ln 2 = \frac{A \ln 2}{\sqrt{3}} = \frac{A \ln 2}{4\sqrt{3}\pi\gamma\ell_P^2}.$$

Involves γ, which can be chosen to yield Bekenstein-Hawking entropy

$$\frac{A}{4\ell_P^2} \Rightarrow \gamma = \frac{\ln 2}{\sqrt{3}\pi}.$$

To implement m constraint, 2^n states written as

$$2^n = 1 +$$

(all spins up) +

For zero total projection, $m = 0$, $n/2$ are up.

If n odd, no such state,

but if n is even, number of states $= {}^nC_{n/2}$.

For large n, Stirling approximation:

$$\begin{aligned} \ln n! &\simeq \ln[\sqrt{2\pi n}(\frac{n}{e})^n] \\ &= n \ln n - n + \frac{1}{2}\ln(2\pi n), \end{aligned}$$

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(all spins up) + (1 down)+

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$$\frac{A}{4\ell_P^2} \Rightarrow \gamma = \frac{\ln 2}{\sqrt{3}\pi}.$$

To implement m constraint, 2^n states written as

$$2^n = 1 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n :$$

(all spins up) + (1 down)+ (2 down)+...+ (all spins down)
For zero total projection, $m = 0$, $n/2$ are up.
If n odd, no such state,
but if n is even, number of states $= {}^nC_{n/2}$.
For large n, Stirling approximation:

$$\begin{aligned} \ln n! &\simeq \ln[\sqrt{2\pi n}(\frac{n}{e})^n] \\ &= n \ln n - n + \frac{1}{2}\ln(2\pi n), \end{aligned}$$

$$\ln \frac{n!}{(n/2)!(n/2)!} \simeq n \ln 2 - \frac{1}{2} \ln n + \frac{1}{2} \ln 2 - \frac{1}{2} \ln \pi.$$

If n independent piece neglected for large n, entropy is

$$\frac{A}{4\ell_P^2} - \frac{1}{2} \ln A.$$

If in addition one wants *total ang mom to vanish*, no. of states with total projection 1 must be subtracted:

$${}^nC_{n/2} - {}^nC_{n/2+1} = {}^nC_{n/2}(1 - \frac{n/2}{n/2+1}) \Rightarrow \frac{A}{4\ell_P^2} - \frac{3}{2} \ln A.$$

4.3. General spin

So far only $j = 1/2$ spins at each puncture.

If spin j at all punctures, area A needs $n = A/[2\sqrt{j(j+1)}]$ punctures.

No. of states $(2j+1)^n$ if m constraint neglected

$$\Rightarrow n \ln(2j+1) = A \ln(2j+1)/[2\sqrt{j(j+1)}].$$

Decreases with increasing j (because $\ln(2j+1)$ increases slowly compared with $\sqrt{j(j+1)}$)
Higher spins contribute less to entropy.

General configuration: s_j punctures with spin j

$$\Rightarrow N = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j (2j+1)^{s_j}$$

if m constraint neglected
(first factor gives number of ways of choosing location of spins, second factor counts no. of spin states at punctures)
Sum N over all nonnegative s_j consistent with given A.
Estimate sum by maximizing $\ln N$ w.r.t. s_j subject to fixed A.

Use Stirling again, neglecting last piece

$$\ln N = \sum_j s_j \ln \frac{2j+1}{s_j} + (\sum_j s_j) \ln(\sum_j s_j),$$

$$\delta \ln N = \sum_j \delta s_j \Big[\ln(2j+1) - \ln s_j + \ln \sum_k s_k \Big],$$

With some Lagrange multiplier λ to implement area constraint,

$$\ln(2j+1) - \ln s_j + \ln \sum_k s_k = \lambda \sqrt{j(j+1)}.$$

$$s_j = (2j+1) \exp \Big[-\lambda \sqrt{j(j+1)} \Big] \sum_k s_k \ .$$

Sum over j:

$$\sum_j (2j+1) \exp \Big[-\lambda \sqrt{j(j+1)} \Big] = 1,$$

which determines $\lambda \approx 1.72$.

$$\ln N = \lambda A/2.$$

To make this $\frac{A}{4\ell_P^2}$ $(4\pi\gamma\ell_P^2 = 1)$ Barbero-Immirzi

$$\gamma = \lambda/(2\pi) \approx 0.274.$$

Sum over s_j may raise this value, projection constraint lowers it.

Alternative:

$$\sum_j 2\exp[-\tilde{\lambda}\sqrt{j(j+1)}] = 1$$

Difference in counting procedures related to whether j is to be counted.

States with same m on punctures but different j regarded above as different, alternative approach considers them same: fewer states.

m defined in 'horizon Hilbert space' while j defined in 'volume Hilbert space'.

However, area of horizon involves j!

4.4. Logarithmic correction

Impose constraint of zero angular momentum projection.

$s_{j,m}$: punctures carrying spin j and projection m.

$s_j = \sum_m s_{j,m}$ involved in area constraint, while $\sum_{j,m} m s_{j,m} = 0$.

Total number of ways of distributing these spins

$$N_{\rm cor} = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j \frac{s_j!}{\prod_m s_{j,m}!} = \frac{(\sum_{j,m} s_{j,m})!}{\prod_{j,m} s_{j,m}!}.$$

Extremize variation of $\ln N_{\rm cor}$ with two Lagrange multipliers λ, α to implement constraints,

$$-\ln s_{j,m} + \ln \sum_{k,n} s_{k,n} = \lambda\sqrt{j(j+1)} + \alpha m,$$

$$\frac{s_{j,m}}{\sum_{k,n} s_{k,n}} = \exp[-\lambda\sqrt{j(j+1)} - \alpha m].$$

Projection constraint requires $\sum m \exp[-\alpha m] = 0$, i.e., α=0, distribution same as before.

To estimate **sum** over $s_{j,m}$ configurations, approximate sums by integrals.
To study variation of $\ln N_{\rm cor}$ with $s_{j,m}$, note that

$$\begin{aligned}\ln(s+\delta s)! &\simeq (s+\delta s)\ln(s+\delta s)-(s+\delta s)\\ &\simeq s\ln s - s + (\ln s)\delta s + (\delta s)^2/(2s).\end{aligned}$$

Terms linear in δs cancel out because of extremization, quadratic part on exponentiation leads to factors of

$$\exp[-(\delta s_{j,m})^2/(2s_{j,m})].$$

Factors of $1/\sqrt{2\pi s_{j,m}}$ from Stirling cancelled by $\sqrt{2\pi s_{j,m}}$ from gaussian integration:

$$\int_{-\infty}^{\infty} d(\delta s_{j,m}) \exp\Big[-\frac{(\delta s_{j,m})^2}{2s_{j,m}}\Big] = \sqrt{2\pi s_{j,m}}.$$

Each $\sqrt{s_{j,m}} \propto \sqrt{A}$.
Area constraint and projection constraint reduce number of summations, hence reduce number of factors of $\sqrt{A}$ by **two**.
But numerator has extra factor $(\sum s_{j,m})^{1/2} \propto \sqrt{A}$.
Net factor $1/\sqrt{A}$ survives, entropy

$$\ln \sum N_{\rm cor} \simeq \lambda A/2 - \frac{1}{2} \ln A.$$

Same log correction as for spin $1/2$.

4.5. Correction to linearity

Linearity of S with A not fully borne out by numerical investigation.
Can be understood by realizing nature of area constraint.

$$A = 2 \sum_j s_j \sqrt{j(j+1)} = [s_{1/2}\sqrt{3} + 2 s_1 \sqrt{2} + s_{3/2}\sqrt{15} + \ldots],$$

so if natural numbers s_j change, $s_{1/2}, s_1, s_{3/2}, \ldots$ cannot mix,
but some mixing possible:

$$
\begin{aligned}
A = \quad & [s_{1/2}\sqrt{3} + 4s_3\sqrt{3} + 15s_{25/2}\sqrt{3} + \ldots] \\
+ \quad & [2s_1\sqrt{2} + 12s_8\sqrt{2} + 70s_{49}\sqrt{2} + \ldots] + \cdots
\end{aligned}
$$

each set must be separately constant.

Sets of compatible spins $N_1 \equiv \{1/2, 3, 25/2, ...\}, N_2 \equiv \{1, 8, 49, ...\}, ...$

There may be several constraints, *number depends on area*

$$A = \sum_N A_N \equiv 2 \sum_N \sum_{j \in N} s_j \sqrt{j(j+1)}.$$

Corresponding Lagrange multipliers λ_N satisfy

$$\sum_N \sum_{j \in N} (2j+1) \exp\Big[-\lambda_N \sqrt{j(j+1)}\Big] = 1,$$

$$S = \sum_N \lambda_N A_N/2 = \bar{\lambda} A/2,$$

$\bar{\lambda} \leq 1.72$, depends on ratio $A_1 : A_2 : \cdots$.

4. Loop quantum gravity

From the seventies

Not so old

Recent work

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