

ELECTROSTATIC FIELD IN MATTER

1. *Introduction to polar and non-polar molecules and the effect of electric field on them.*
2. *Effect of electric field on neutral atoms.*
3. *Polarizability and the field of polarized object.*
4. *Bound charges and their physical interpretation.*
5. *Electric Displacement.*
6. *Few examples.*

Dielectrics

1. Dielectrics are substances that do not conduct electric current. They do not have free electric charges.
2. Air, glass, mica and paraffin wax are the examples of dielectric.
3. The resistivity of the dielectric is of the order of 10^{22} ohm-cm (resistivity of conductors is of the order of 10^{-6} ohm-cm).
4. The electric charge given to a dielectric remains fixed in the region of application in contrasts to the conductors where the charge immediately spreads over the surface.

Classification of dielectrics:

Molecules have equal amount of positive and negative charges.

Set of positively charged point particles are embedded in a set of negative charge distribution.

Center of positive charges and Center of negative charges exist in the molecules.

Polar Molecules:

→ Center of gravity of positive charges **do not** coincide with the center of gravity of negative charge.

Example: H_2O , CO_2 , NH_3 , HCl etc. → polar dielectrics.

→ Have net dipole moment.

Non-polar Molecules:

→ Center of gravity of positive charges coincides with the center of gravity of negative charges.

Example: H_2 , N_2 , O_2 , Cl_2 etc.

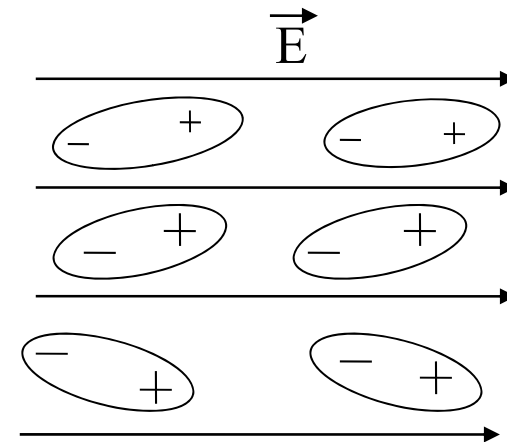
→ Non-polar molecules do not have any permanent dipole moment.

Effect of electric field on polar molecules:

→ In the absence of electric field, the dipole moment of these polar molecules point in the random directions and arrange themselves in a closed chain and hence the net dipole moment is zero.

→ When an external electric field is applied, the chains are broken and those molecules align themselves parallel to the direction of the applied electric field, but the alignment may not be complete due to the thermal agitation of the molecule.

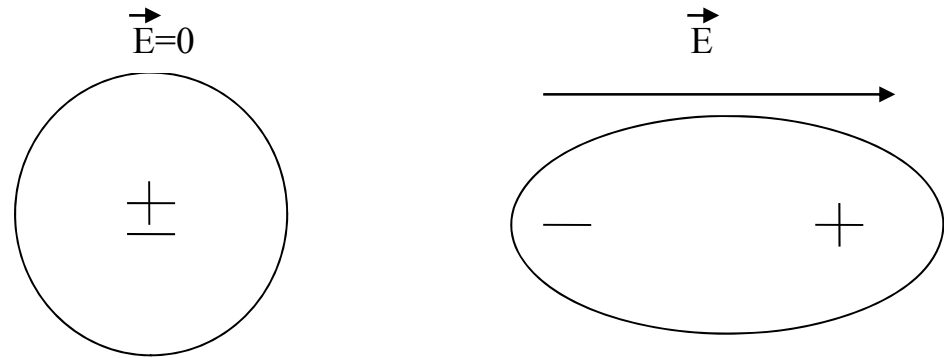
As the strength of the electric field is increased and if the temperature is decreased, complete alignment is possible



Effect of electric field on non-polar molecules:

→ Electric field is applied across the non-polar molecule, the centre of negative and positive charges suffer a displacement and the molecule acquires a temporary polar character by induction.

→ Gets induced dipole moment



→ When the applied field is removed, the induced dipole moment disappears and molecule again becomes a non-polar.

→ The alignment is proportional to the strength of the external electric field but almost independent of temperature.

What happens to a neutral atom when it is placed in an electric field?

First guess might be:

Absolutely nothing – since the atom is not charged, the field has no effect on it

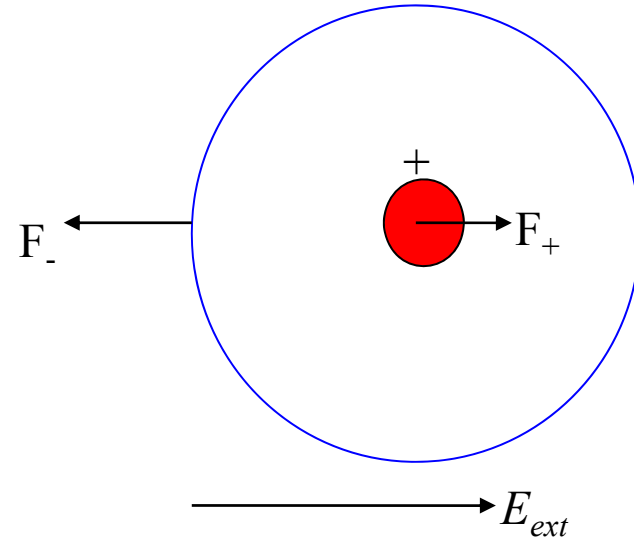
But this is incorrect

Though the atom as a whole is neutral, the positive charge is concentrated in the nucleus (radius = 10^{-14} m) while the negative charge forms an electron cloud (radius = 10^{-10} m) surrounding the nucleus

These two regions of charge within the atom are influenced by the field:

1. The nucleus experiences a force pointing in the same direction as the external electric field.
2. The negatively charged electron cloud experiences a force of the same magnitude, but in the opposite direction to that of the electric field.

Atom in an external electric field



Result?

- Nucleus and Center of the negative charges separate
- Distance between them is balanced by the electrostatic attractive force and external applied electric field
- Atom gets **polarized**.

One can treat the electron cloud as a constant volume charge density ρ , radius a and total charge as $-q$.

$$\rho = \frac{-q}{\frac{4}{3}\pi a^3} = -\frac{3q}{4\pi a^3}$$

The electric field inside the uniformly charged cloud is equal to

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{qr}{a^3}$$

where r is distance from the center of the cloud. Suppose that as the result of the external electric field the nucleus moves a distance d with respect to the center of the cloud. The electric force exerted on the nucleus by the electron cloud is equal to

$$F_{cloud} = qE(d) = -\frac{1}{4\pi\epsilon_0} \frac{q^2 d}{a^3}$$

The force balance equation is

$$\overline{F}_{cloud} + \overline{F}_{ext} = 0$$

$$qE_{ext} - \frac{1}{4\pi\epsilon_0} \frac{q^2 d}{a^3} = 0$$

The equilibrium distance d is thus equal to $d = 4\pi\epsilon_0 a^3 \frac{E_{ext}}{q}$

The induced dipole moment of the atom is $\vec{p} = q\vec{d} = 4\pi\epsilon_0 a^3 \vec{E}_{ext}$

$$\vec{p} = \alpha \vec{E}_{ext}$$

α is called atomic polarizability  $\alpha = 4\pi\epsilon_0 a^3$

In a material, there are many atoms as well as voids. Instead of treating each atoms separately, one can take a small volume and evaluate the net dipole moment.

For macroscopic purposes – dipole moment per unit volume

$$\vec{P} = \vec{p} / Volume$$

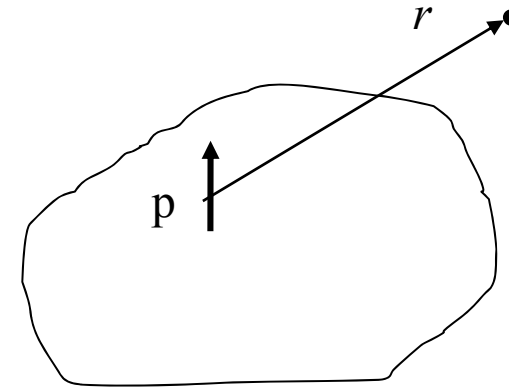
POTENTIAL DUE TO A POLARIZED OBJECT

Suppose we have a piece of a polarized material (that is, an object containing a lot of microscopic dipoles lined up) with dipole moment per unit volume equal to $\vec{P}$. The electrostatic potential generated by a single dipole

$$V = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

In the present case dipole moment $\vec{p} = \vec{P} d\tau$ in each volume element $d\tau$, so the total potential is

$$V = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{\vec{P} \cdot \hat{r}}{r^2} d\tau$$



We know that

$$\nabla\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$$

$$\therefore V = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \bar{P} \bullet \nabla\left(\frac{1}{r}\right) d\tau$$

Using the following relation (product rule for the vector operation)

$$\bar{\nabla} \bullet \left(\frac{1}{r} \bar{P}\right) = \frac{1}{r} (\bar{\nabla} \bullet \bar{P}) + \bar{P} \bullet \bar{\nabla}\left(\frac{1}{r}\right)$$

We can rewrite the expression for V as

$$V = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \bar{\nabla} \bullet \left(\frac{1}{r} \bar{P}\right) d\tau - \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{1}{r} (\bar{\nabla} \bullet \bar{P}) d\tau$$

Using divergence theorem

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \int_{\text{surface}} \frac{1}{r} \bar{P} \bullet d\bar{a} - \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{1}{r} (\bar{\nabla} \bullet \bar{P}) d\tau \\ &= \frac{1}{4\pi\epsilon_0} \int_{\text{surface}} \frac{1}{r} \sigma_b da + \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{1}{r} \rho_b d\tau \end{aligned}$$

where

$$\sigma_b = \bar{P} \bullet d\hat{n} \quad \text{Surface bound charge density}$$

$$\rho_b = -(\bar{\nabla} \bullet \bar{P}) \quad \text{Volume bound charge density}$$

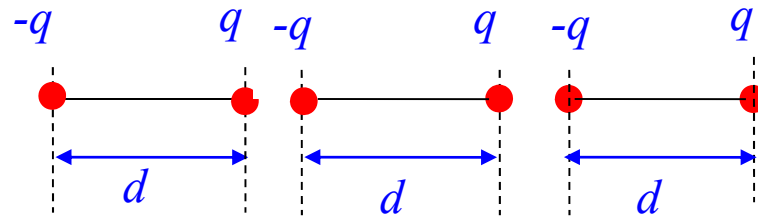
Here the unit vector n is outward normal.

Potential (and also the electric field) generated by the polarized object

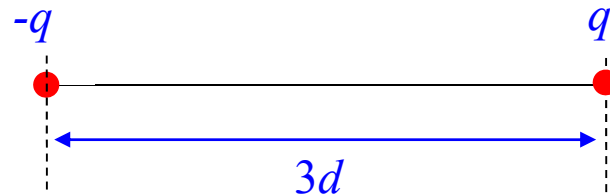
→ Due to the surface and volume bound charges

Physical interpretation of bound charge

The bound charges just introduced are not just mathematical artifacts, but are real charges, bound to the individual dipoles of the material. Consider for example the three dipoles shown in figure below.



When they are aligned (lengthwise) the center of charges cancel, and the system looks like a single dipole with dipole moment $3dq$ as shown in figure below



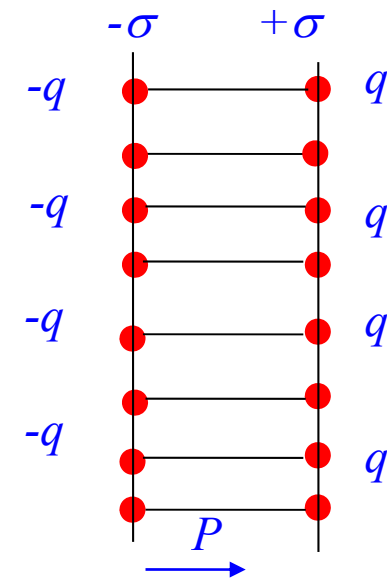
In a uniformly polarized material of thickness s and polarization $\overline{P}$ all the dipoles are perfectly aligned as shown in figure below. The net result of the alignment of the individual dipoles is a positive surface charge on one side of the material and negative surface on the opposite side.

Consider a cylinder with surface area A whose axis is aligned with the direction of polarization of a polarized material. The total dipole moment of this cylinder is equal to

$$\overline{P}_{cylinder} = As\overline{P}$$

Since the only charge of the system reside on the end cups of the cylinder (volume charges cancel in a uniformly polarized materials), the net charge there must be equal to

$$q_{end} = \frac{q_{cylinder}}{S} = AP$$



The charge density on the surface is therefore equal to

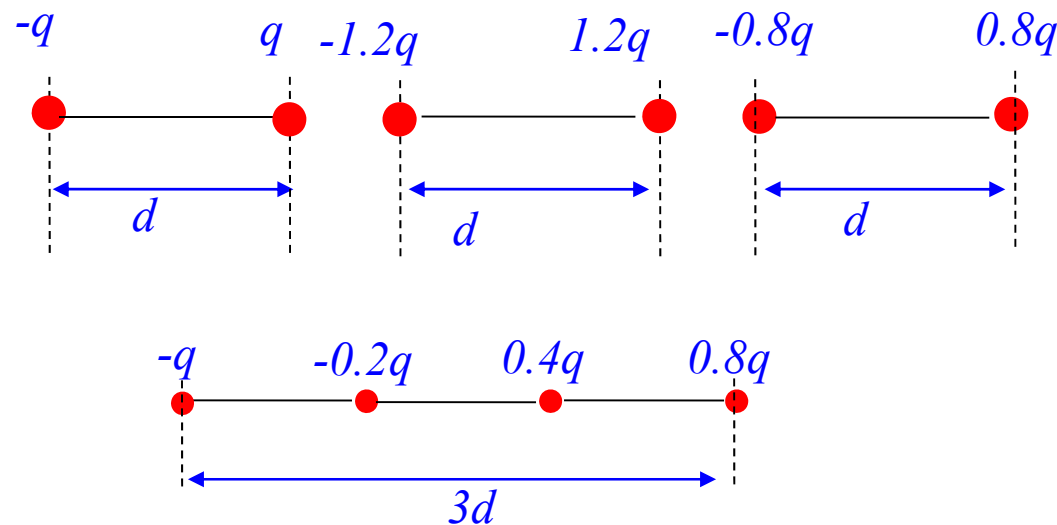
$$\sigma = \frac{q_{end}}{A} = P$$

If the surface of the material is not perpendicular to the direction of the polarization then surface charge density will be less than P (surface charge distributed over the large area) and equal to

$$\sigma = \vec{P} \bullet \hat{n}$$

Where $\hat{n}$ is the unit vector perpendicular to the surface of the material., pointing $\hat{n}$ outwards. For the material shown in last figure this equation immediately shows that a positive surface charge resides on the right surface ($\vec{P}$ parallel to $\hat{n}$) and the negative surface charge resides on the left surface ($\vec{P}$ anti parallel to $\hat{n}$). Since these charges reside on the surface and are bound to the dipoles they are called as the bound surface charge or σ_b

If the material is uniformly polarized then volume charge density is equal to zero. However, if the polarization is not uniform then there will be a net volume charge inside the material. Consider a system of three aligned dipoles as shown in figure below.



If the polarization is not uniform then the strength of the individual dipole will vary. Assuming that the physical size (length) of the dipole shown in the figure above is same, then varying dipole strength is the result of variations in charge on the ends of dipole

Net charge on the polarization material is zero

→ Total bound charges must also be zero


$$\int_{\text{surface}} \sigma_b da + \int_{\text{volume}} \rho_b d\tau = 0$$


$$\int_{\text{volume}} \rho_b d\tau = - \int_{\text{surface}} \sigma_b da = - \int_{\text{surface}} \bar{P} \cdot da = - \int_{\text{volume}} (\bar{\nabla} \cdot \bar{P}) d\tau$$

Example: The sphere of radius R carries a polarization

$$\vec{P}(\vec{r}) = k\vec{r}$$

Where k is the constant and r is the vector from the center

1. Calculate bound charges σ_b and ρ_b .
2. Find the field inside and outside the sphere.

The unit vector $\hat{n}$ on the surface of the sphere is equal to the radial unit vector. The bound surface charge is equal to

$$\sigma_b = \vec{P} \cdot \hat{n} \Big|_{r=R} = k\vec{r} \cdot \hat{r} \Big|_{r=R} = kR$$

The bound volume charge equal to

$$\rho_b = -(\nabla \cdot \vec{P}) = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 kr) = -3k$$

2. First consider the region outside the sphere. The electric field in this region due to the surface charge is equal to

$$\overline{E}_{surface}(r) = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^2 \sigma_b}{r^2} \hat{r} = \frac{kR^3}{\epsilon_0 r^2} \hat{r}$$

The electric field in this region due to the surface charge is equal to

$$\overline{E}_{volume}(r) = \frac{1}{4\pi\epsilon_0} \frac{\frac{4}{3}\pi R^3 \rho_b}{r^2} \hat{r} = -\frac{kR^3}{\epsilon_0 r^2} \hat{r}$$

Therefore total electric field outside the sphere is equal to zero

Now consider the region inside the sphere. The electric field due to surface charge is equal to zero. The electric field due to volume charge is equal to

$$\overline{E}_{volume}(r) = \frac{1}{4\pi\epsilon_0} \frac{\frac{4}{3}\pi r^3 \rho_b}{r^2} \hat{r} = -\frac{kr}{\epsilon_0} \hat{r}$$

Example: A dielectric cube of side s , centered at the origin carries a polarization

$$\overline{P}(\overline{r}) = k\overline{r}$$

Where k is the constant. Find all the bound charges, and check that they add up to zero

The bound volume charge density is equal to

$$\rho_b = -(\overline{\nabla} \bullet \overline{P}) = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 k r) = -3k$$

Since the bound volume charge density is constant, the total bound volume charge in a cube is equal to product of the charge density and the volume.

$$q_{\text{volume}} = -3ks^3$$

The surface charge density σ_b is equal to $\sigma_b = \overline{P} \bullet \hat{n} = k\overline{r} \bullet \hat{n}$

$$\vec{r} \cdot \hat{n} = r \cos \theta = \frac{1}{2}s$$

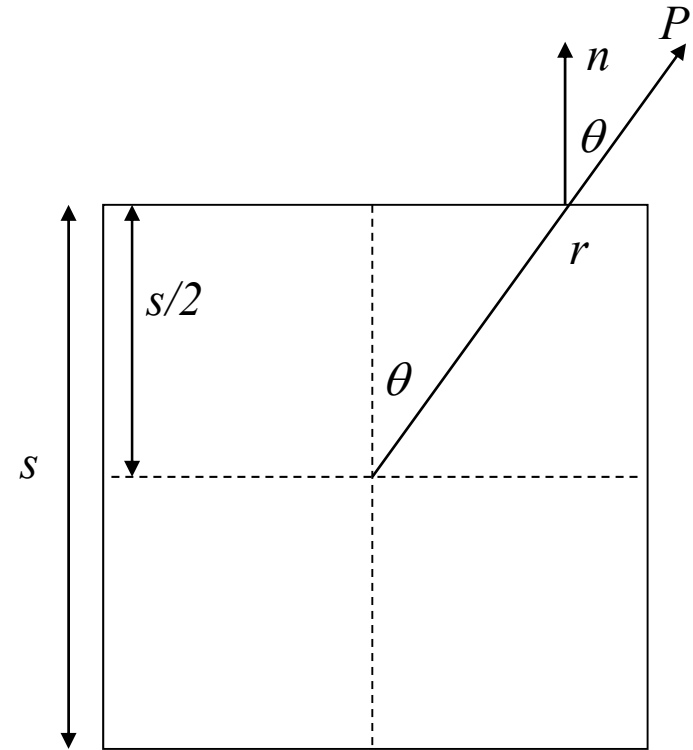
$$\longrightarrow \sigma_b = k\vec{r} \cdot \hat{n} = \frac{1}{2}ks$$

The surface charge density is constant across the surface of the cube

$$\longrightarrow q_{surface} = \left(\frac{1}{2}ks \right) (6s^2) = 3ks^3$$

Total bound charge on the cube is equal to

$$\begin{aligned} q_{total} &= q_{volume} + q_{surface} \\ &= -3ks^3 + 3ks^3 \\ &= 0 \end{aligned}$$



ELECTRIC DISPLACEMENT VECTOR

The electric field generated by a polarized material

→ Due to its bound charges.

If free charges are also present

→ Total electric field produced by this system

→ Due to both bound charges and free charges



Apply Gauss law that includes both free and bound charges

$$\begin{aligned}\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} = \frac{\rho_{bound} + \rho_{free}}{\epsilon_0} \\ &= \frac{1}{\epsilon_0} \left(-\nabla \cdot \vec{P} + \rho_{free} \right)\end{aligned}$$

$$\nabla \bullet (\epsilon_0 \vec{E} + \vec{P}) = \rho_{free}$$

The expression in the parenthesis is called the **electric displacement** D

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Gauss law in materials can be written as

$$\nabla \bullet \vec{D} = \rho_{free} \quad \text{Differential form}$$

and

$$\oint_{surface} \vec{D} \bullet d\vec{a} = Q_{free} \quad \text{Integral form}$$

→ Q_{free} denotes the total free charge enclosed in the volume

→ Makes reference only to free charges that we can control

Although it seems that the displacement $\vec{D}$ has a properties similar to electric field $\vec{E}$ there are some significant differences. For example, the curl of $\vec{D}$ is equal to

$$\nabla \times \vec{D} = \epsilon_0 \nabla \times \vec{E} + \nabla \times \vec{P} = \nabla \times \vec{P}$$

And in general not equal to zero. Since the curl of $\vec{D}$ is not necessarily to zero, there is in general no potential that generates $\vec{D}$.

The famous Helmholtz theorem tell us that if we know the curl and a divergence of a vector function $\vec{v}$ then it is sufficient information to uniquely define the vector function $\vec{v}$. Therefore, the electric field $\vec{E}$ is uniquely defined by Gauss's law since we know the curl of $\vec{E}$ is zero everywhere. The displacement vector $\vec{D}$ on other hand is not uniquely determined by the free charge distribution, but requires additional information (like for example $\vec{P}$)

Example: Suppose the field inside a large piece of dielectric is E_0 , so that the electric displacement is equal to $\vec{D}_0 = \epsilon_0 \vec{E}_0 + \vec{P}$

1. A small spherical cavity is hallowed out of the material. Find the field at the center of the cavity in terms of E_0 and P . Also find the displacement vector at the center of the cavity in terms of D_0 and P .
2. Do the same for needle shaped cavity running parallel to P .
3. Do the same for a thin wafer-shaped cavity perpendicular to P .

Solution-1: Making spherical cavity from the large polarized object is equivalent to superimposing another material with polarization $-\mathbf{P}$.

The electric field inside a sphere with polarization $-\mathbf{P}$ is uniform and given by

$$\vec{E} = -\frac{(-\vec{P})}{3\epsilon_0}$$

The field at the center of the cavity is therefore

$$\vec{E}_{center} = \vec{E}_0 + \vec{E}_{sphere} = \vec{E}_0 + \frac{1}{3\epsilon_0} \vec{P}$$

Therefore, at the center of the cavity,

$$\vec{D}_{center} = \epsilon_0 \vec{E}_{center} = \epsilon_0 \vec{E}_0 + \frac{1}{3} \vec{P} = \vec{D}_0 - \frac{2}{3} \vec{P}$$

2. Similar the earlier problem, needle shaped cavity is nothing but the superposition of needle shaped material with polarization $-\mathbf{P}$. The electric field of a polarized needle of length s is equal to two point charges ($+q$ and $-q$) located a distance s apart. The charge on top of the needle will be negative, while the charge on the bottom of the needle will be positive. The charge density on the end caps of the needle is equal to P . Therefore $q = \sigma_b A = PA$

where A is the surface area of the end caps of the needle. The electric field generated by the needle at its center is:

$$\vec{E}_{needle} = \frac{1}{4\pi\epsilon_0} \frac{(+PA)}{\frac{1}{4}s^2} \hat{e}_z - \frac{1}{4\pi\epsilon_0} \frac{(-PA)}{\frac{1}{4}s^2} \hat{e}_z = \frac{2}{\pi\epsilon_0} \frac{PA}{s^2} \hat{e}_z$$

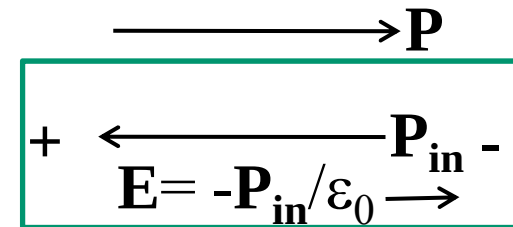
When $A \rightarrow 0$, $\vec{E}_{needle} \rightarrow 0$. Therefore, $\vec{E}_{center} = \vec{E}_0$

The displacement vector at the center is $\vec{D}_{cent.} = \epsilon_0 \vec{E}_{center} = \epsilon_0 \vec{E}_0 = \vec{D}_0 - \vec{P}$

3. The thin wafer shaped cavity has $-\mathbf{P}$. The electric field due to this is

$$\vec{E}_{wafer} = \vec{E}_0 + \frac{\vec{P}}{\epsilon_0}$$

$$\vec{D}_{wafer} = \epsilon_0 \vec{E}_{wafer} = \epsilon_0 \vec{E}_0 + \vec{P} = \vec{D}_0$$



$$+ \left[\overleftarrow{\mathbf{P}_{in}} \right] \mathbf{E} = -\mathbf{P}_{in} / \epsilon_0 \rightarrow -$$

LINEAR DIELECTRICS

A material is called linear material if the polarization is linearly proportional to the electric field. $\vec{P} = \epsilon_0 \chi_e \vec{E}$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$$

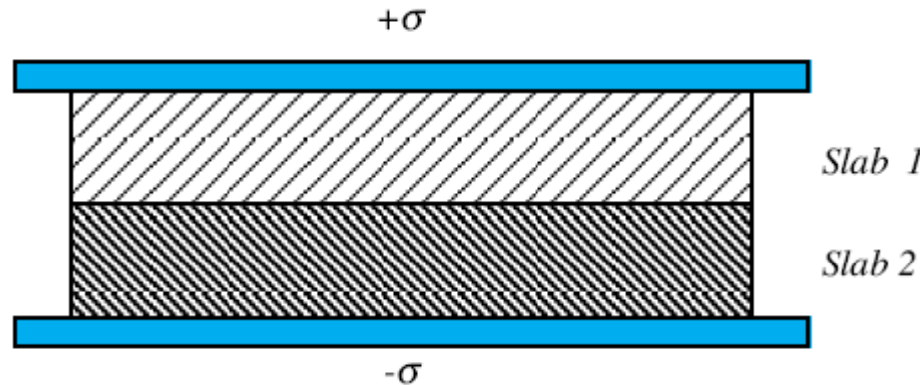
χ_e is the electric susceptibility, ϵ_r is the dielectric constant and ϵ is the dielectric permittivity

If the material is isotropic and homogeneous, there bound to be a relation between free charge density and bound charge density

$$\rho_f = \nabla \cdot \vec{D} = \nabla \cdot \epsilon \vec{E} = \epsilon_0 \epsilon_r \nabla \cdot \vec{E} = \epsilon_0 \epsilon_r \nabla \cdot \frac{\vec{P}}{\epsilon_0 \chi_e} = \frac{\epsilon_r}{\epsilon_r - 1} \nabla \cdot \vec{P} = -\frac{\epsilon_r}{\epsilon_r - 1} \rho_b$$

$$\sigma_b = \vec{P} \cdot \hat{n} = (\vec{D} - \epsilon_0 \vec{E}) \cdot \hat{n} = \left(\vec{D} - \frac{\vec{D}}{\epsilon_r} \right) \cdot \hat{n} = \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \vec{D} \cdot \hat{n} = \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \sigma_f$$

Problem: The space between two parallel plate capacitor is filled with two slabs of linear dielectric material. As shown in figure. Each slab has thickness s , so that total distance between the plates is $2s$.

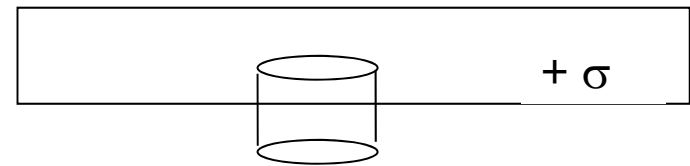


Slab 1 has a dielectric constant ϵ_{r1} and slab 2 has a dielectric constant of ϵ_{r2} . The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.

1. Find the electric displacement D in each slab.
2. Find the electric field E in each slab.
3. Find the polarization in each slab.
4. Find the potential difference between the plates.
5. Find the location and amount of all bound charge.

(1) The electric displacement $\vec{D}_1$ in slab 1 can be calculated by using Gauss's law. Consider a cylinder with cross sectional area A and axis parallel to z -axis being used as a Gaussian surface. The top of the cylinder is located inside the top of the metal plate (where the electric displacement is zero) and the bottom of the cylinder is located inside the dielectric slab 1.

Apply Gauss's law $\int \vec{D} \cdot d\vec{a} = Q_{f \text{ enclosed}}$



To the Gaussian surface, we get $D A = \sigma A$

Note $\vec{D}=0$ in the metal plate.

$$\longrightarrow D = \sigma$$

Similarly for the second slab $D = -\sigma$

(2) The electric field in slab 1 is: $\vec{E}_1 = \frac{1}{\epsilon_{r1}\epsilon_0} \vec{D}_1 = \frac{\sigma}{\epsilon_{r1}\epsilon_0} (-\hat{e}_z)$

Similarly we obtain for slab 2: $\vec{E}_2 = \frac{1}{\epsilon_{r2}\epsilon_0} \vec{D}_2 = \frac{\sigma}{\epsilon_{r2}\epsilon_0} (-\hat{e}_z)$

(3) Once $\vec{D}$ and $\vec{E}$ are known, the polarization $\vec{P}$ can be calculated

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

Therefore in Slab 1 $\vec{P}_1 = \epsilon_0 \chi_{e1} \frac{\vec{D}_1}{\epsilon_0 \epsilon_{r1}} = \left(\frac{\chi_{e1}}{\epsilon_{r1}} \right) \sigma (-\hat{e}_z) = \left(\frac{\epsilon_{r1} - 1}{\epsilon_{r1}} \right) \sigma (-\hat{e}_z)$

in Slab 2: $\vec{P}_2 = \epsilon_0 \chi_{e2} \frac{\vec{D}_2}{\epsilon_0 \epsilon_{r2}} = \left(\frac{\chi_{e2}}{\epsilon_{r2}} \right) \sigma (-\hat{e}_z) = \left(\frac{\epsilon_{r2} - 1}{\epsilon_{r2}} \right) \sigma (-\hat{e}_z)$

(4) The potential difference between the top plate and the bottom plate is equal to $\Delta V = V_{top} - V_{bottom} = E_1 s + E_2 s$

 $\Delta V = - \left(\frac{\sigma}{\epsilon_{r1} \epsilon_0} + \frac{\sigma}{\epsilon_{r2} \epsilon_0} \right) s = - \frac{\sigma s}{\epsilon_0} \left(\frac{1}{\epsilon_{r1}} + \frac{1}{\epsilon_{r2}} \right)$

5. There are no volume bound charges since $\mathbf{P}$ is constant.

σ_b at the bottom and top of slab 1?

σ_b at the bottom and top of slab 2?

Problem: A sphere of linear dielectric material has embedded in it a uniform free charge density ρ . Find the potential at the center of the sphere, if its radius is R and its dielectric constant ϵ_r

Solution: The system has a spherical symmetry and therefore the electric displacement $\mathbf{D}$ is easy to calculate since

$$\nabla \cdot \mathbf{D} = \rho_{free} \quad \nabla \times \mathbf{D} = 0$$

The calculation of $\mathbf{D}$ is very similar to the calculation of $\mathbf{E}$ using Gauss's Law

$$\vec{D} = \frac{1}{4\pi r^2} Q_{free, encl} = \frac{\rho R^3}{3r^2} \quad (r > R)$$

$$\vec{D} = \frac{1}{4\pi r^2} Q_{free, encl} = \frac{1}{3} \rho r \quad (r \leq R)$$

The corresponding electric field is equal to $\vec{E} = \frac{\vec{D}}{\epsilon_r \epsilon_0} = \frac{\rho R^3}{3\epsilon_0 r^2} \quad (r > R)$

$$\vec{E} = \frac{\vec{D}}{\epsilon_r \epsilon_0} = \frac{1}{3\epsilon_r \epsilon_0} \rho r \quad (r \leq R)$$

The potential at the center of the sphere can be calculated by using this electric field as

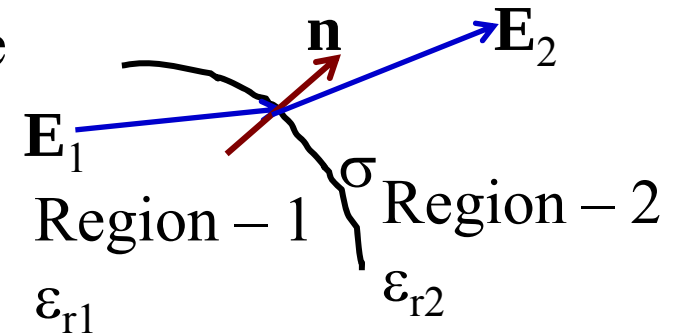
$$\begin{aligned} V(r) &= -\int_{\infty}^0 \vec{E} \bullet d\vec{l} = -\int_{\infty}^R \frac{\rho R^3}{3\epsilon_0 r^2} dr - \int_R^0 \frac{1}{3\epsilon_r \epsilon_0} \rho r dr \\ &= \frac{\rho R^3}{3\epsilon_0 r} \Big|_{\infty}^R - \frac{1}{6\epsilon_r \epsilon_0} \rho r^2 \Big|_R^0 = \frac{\rho R^2}{3\epsilon_0} \left\{ 1 + \frac{1}{2\epsilon_r} \right\} \end{aligned}$$

BOUNDARY CONDITIONS

The boundary between two dielectrics has free surface charge density σ

$$\oint \vec{D} \cdot d\vec{a} = (D_{n2} - D_{n1})A = \sigma A$$

$$\longrightarrow (D_{n2} - D_{n1}) = \sigma$$



$$\oint \vec{E} \cdot d\vec{l} = (E_{t2} - E_{t1})l = 0 \longrightarrow E_{t2} = E_{t1}$$

$$\oint \vec{E} \cdot d\vec{a} = (E_{n2} - E_{n1})A = \frac{\sigma_{total} A}{\epsilon_0}$$

→ If any one is a conductor, then $E_{t1} = E_{t2} = 0$

→ If $\sigma = 0$, then $D_{n2} = D_{n1} \rightarrow \epsilon_{r1} E_{n1} = E_{n2} \epsilon_{r2}$

→ In the charge free boundary, in terms of the angle:

$$\epsilon_2 E_2 \cos \theta_2 = \epsilon_1 E_1 \cos \theta_1$$

$$E_2 \sin \theta_2 = E_1 \sin \theta_1$$



$$\frac{\tan \theta_2}{\tan \theta_1} = \frac{\epsilon_2}{\epsilon_1}$$

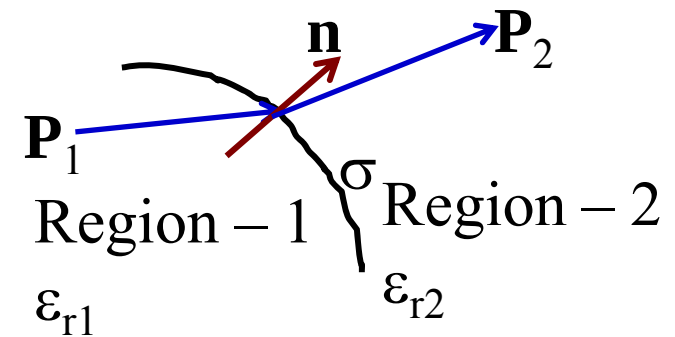
If the media have polarizations P_1 and P_2 ?

$$(D_{n2} - D_{n1}) = \sigma = \epsilon_0 E_{n2} + P_{n2} - \epsilon_0 E_{n1} - P_{n1}$$

$$\sigma = \epsilon_0 \left(\frac{\sigma_b + \sigma}{\epsilon_0} \right) E_{n2} + P_{n2} - P_{n1}$$



$$(P_{n2} - P_{n1}) = -\sigma_b$$



→ Normal component of $\mathbf{D}$ is discontinuous by an amount equal to the surface charge density σ .

→ Normal component of the displacement vector $\mathbf{D}$ is continuous across the charge free boundary between two dielectrics.

→ Tangential component of the electric field is always continuous at any boundary

→ Normal component of the polarization is discontinuous by an amount equal to the negative value of the surface bound charge density

Energy in dielectric systems

Consider a capacitor with capacitance C and charged up to potential V . The total energy stored in the capacitor is equal to the work done during the charging process:

$$W = \frac{1}{2}CV^2$$

If the capacitor is filled with a linear dielectric (with dielectric constant ϵ_r) then total capacitance will increase the vacuum value by a factor K

$$C = \epsilon_r C_{vac}$$

And consequently the energy stored in the capacitor (when held at constant potential) is increased by a factor ϵ_r

A general expression for the energy of the capacitor with dielectric material present can be found by studying the charging process in detail.

Consider a free charge held at potential V . During the charging process the free charge is increased by ρ_{free} . The work done on extra free charge is equal to

$$\Delta W = \frac{1}{2} \int_{volume} \rho_{free} V d\tau$$

Since the divergence of the electric displacement is equal to the free charge density, the divergence of $\vec{D}$ is equal to ρ_{free} . Therefore

$$\Delta W = \frac{1}{2} \int_{volume} [\nabla \cdot \vec{D}] V d\tau$$

Use

$$\nabla \cdot (V\vec{D}) = [\nabla \cdot \vec{D}]V + (\nabla V) \cdot \vec{D}$$

We can rewrite the expression for ΔW as

$$\Delta W = \frac{1}{2} \int_{\text{volume}} \left[\nabla \cdot (V \vec{D}) \right] d\tau + \frac{1}{2} \int_{\text{volume}} \left[(\nabla V) \cdot (\vec{D}) \right] d\tau$$

The first term on right hand side of this equation can be rewritten as

$$\int_{\text{volume}} \left[\vec{\nabla} \cdot (V \vec{D}) \right] d\tau = \int_{\text{surface}} (V \vec{D}) \cdot d\vec{a}$$

Integrating over all space, the surface integral vanishes and therefore

$$\begin{aligned} \Delta W &= \frac{1}{2} \int_{\text{allspace}} \left[\nabla V \cdot \vec{D} \right] d\tau \\ &= \frac{1}{2} \int_{\text{allspace}} \left[\vec{E} \cdot \vec{D} \right] d\tau \end{aligned}$$

This is true for any material

Assuming that the material present in the system are linear dielectrics then

$$\vec{D} = \epsilon \vec{E}$$

By using this relation and rearrangement, The expression for W can finally be rewritten as

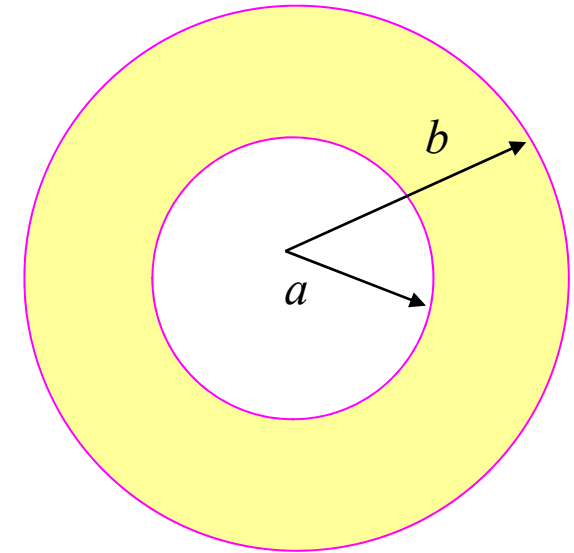
$$W = \frac{1}{2} \epsilon \int_{allspace} E^2 d\tau$$

Example: A spherical conductor of radius a carries a charge Q . It is surrounded by linear dielectric material of susceptibility χ_e , out to a radius b . Find the energy of this configuration.

Solution: Since the system has spherical symmetry, the electric displacement is completely determined by the free charges. It is equal to

$$\vec{D} = \frac{1}{4\pi r^2} Q_{encl} \hat{e}_r = \vec{0} \quad : r \leq a$$

$$\vec{D} = \frac{1}{4\pi r^2} Q_{encl} = \frac{1}{4\pi} \frac{Q}{r^2} \hat{e}_r \quad : r > a$$



$$\vec{E} = 0 \quad \text{for } r \leq a \quad ; \vec{E} = \frac{Q}{4\pi\epsilon} \frac{\hat{e}_r}{r^2} \quad \text{for } a < r \leq b \quad ; \vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{e}_r}{r^2} \quad \text{for } b < r$$

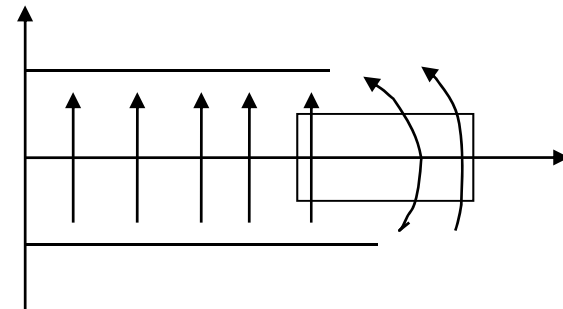
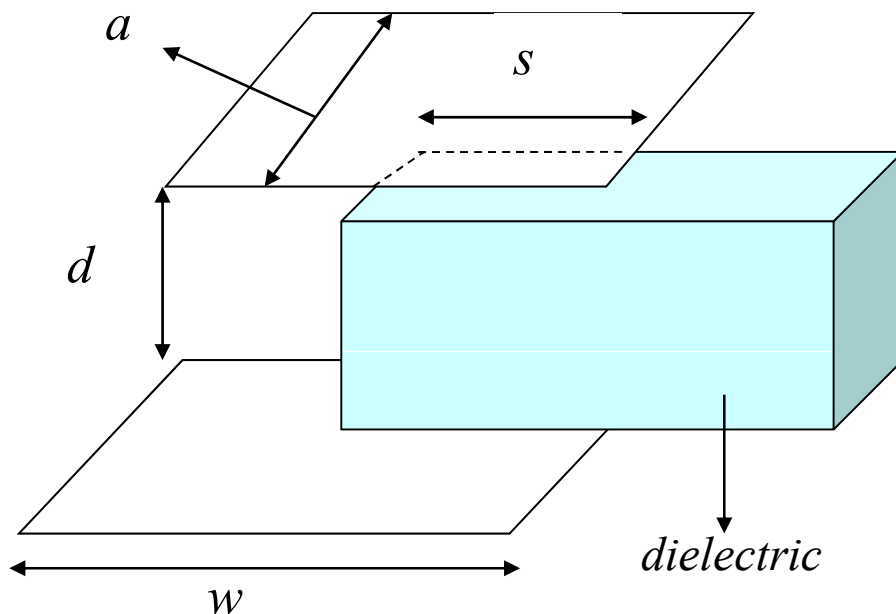
$$\epsilon = \epsilon_0 (1 + \chi_e)$$

$$W = \frac{1}{2} \int_{all} \vec{D} \bullet \vec{E} \, d\tau = \frac{1}{2} 4\pi \int_a^\infty \vec{D} \bullet \vec{E} r^2 dr = 2\pi \int_a^b \frac{1}{16\pi^2 \epsilon} \frac{Q^2}{r^4} r^2 dr + 2\pi \int_b^\infty \frac{1}{16\pi^2 \epsilon_0} \frac{Q^2}{r^4} r^2 dr$$

$$\begin{aligned}
W &= 2\pi \left(\frac{Q^2}{16\pi^2} \right) \left[\left\{ \frac{1}{\varepsilon} \int_a^b \frac{r^2}{r^4} dr \right\} + \left\{ \frac{1}{\varepsilon_o} \int_b^\infty \frac{r^2}{r^4} dr \right\} \right] \\
&= \frac{Q^2}{8\pi\varepsilon_0} \left\{ \frac{1}{(1+\chi_e)} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{b} \right\} = \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)} \left\{ \frac{1}{a} + \frac{\chi_e}{b} \right\}
\end{aligned}$$

Forces on dielectrics

A dielectric slab placed partly between the plates of a parallel-plate capacitor will be pulled inside the capacitor. This force is a result of the fringing fields around the edges of the parallel-plate capacitor (see Figure). **Note: the field outside the capacitor can not be zero** since otherwise the line integral of the electric field around a closed loop, partly inside the capacitor and partly outside the capacitor, would not be equal to zero.



A direct calculation of this force requires a knowledge of the fringing fields of the capacitor which are often not well known and difficult to calculate. An alternative method that can be used is to determine this force is to calculate the change in the energy of the system when the dielectric is displaced by a distance ds . The work to be done to pull the dielectric out by an infinitesimal distance ds is equal to

$$dW = \vec{F}_{us} \cdot d\vec{s}$$

Where F_{us} is the force provided by us to pull the slab out of the capacitor. This force must be equal in magnitude but directed in a direction opposite to the force exerted by the electric field on the slab. Thus

$$\left| \vec{F}_{field} \right| = - \left| \vec{F}_{us} \right| = - \frac{dW}{ds}$$

Consider the situation shown in Figure where the slab of dielectric is inserted to a depth s in the capacitor. The capacitance of this system is equal to

$$\begin{aligned}
 C &= C_{vac} + C_{diel} = \frac{\epsilon_0(w-s)a}{d} + \frac{\epsilon_r\epsilon_0sa}{d} \\
 &= \frac{\epsilon_0a}{d} [w - s + \epsilon_r s] \\
 &= \frac{\epsilon_0a}{d} [w + s(\epsilon_r - 1)] \\
 &= \frac{\epsilon_0a}{d} [w + \chi_e s]
 \end{aligned}$$

If the total charge on the top plate is Q then the energy stored in the capacitor is equal to

$$W = \frac{1}{2} \frac{Q^2}{C}$$

The force on the dielectric can now be calculated and is equal to

$$F_{field} = -\frac{dW}{ds} = \frac{1}{2} \frac{Q^2}{C^2} \frac{dC}{ds}$$

But

$$\frac{dC}{ds} = \frac{\epsilon_0 \chi_e a}{d}$$

Therefore

$$F_{field} = \frac{1}{2} \frac{Q^2}{C^2} \frac{\epsilon_0 \chi_e a}{d} = \frac{1}{2} \frac{\epsilon_0 \chi_e a}{d} V^2$$

Problem 4.29: Two long coaxial cylindrical metal tubes (inner radius a , outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e , mass density ρ). The inner one is maintained at potential V , and the outer one is grounded. To what height h does the oil rise in the space between the tubes?