Department of Physics
Indian Institute of Technology, Madras
PH5840 Quantum Computation and Quantum Information

Assignment 1 | 17 August 2017 (Due: 28 August) | 25 Marks

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(1) Composite systems and the tensor product [5 Marks]

Recall that the *Kronecker product* $A \otimes B$ for a pair of matrices A, B is defined as follows:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mm}B \end{pmatrix},$$

where $(A)_{ij} = a_{ij}$ is an $m \times m$ matrix and $(B)_{ij} = b_{ij}$ is a $n \times n$ matrix.

(i) Evaluate the eigenvalues of $\sigma_Z \otimes \sigma_Z$, and express the corresponding eigenstates as tensor products of the states $|0\rangle$, $|1\rangle$.

(ii) If A and B are Hermitian operators, is $A \otimes B$ also Hermitian?
Hint: Does the adjoint (complex conjugate transpose) operation distribute over the tensor product, i.e. is $(A \otimes B)^\dagger$ the same as $A^\dagger \otimes B^\dagger$?

(iii) Recall that the *Hadamard* operation on a single qubit is defined as

$$H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|).$$

Write out the explicit matrix representation for $H^{\otimes 2}$, the Hadamard operation on two qubits. Show that the Hadamard operation on n qubits can be written as

$$H^{\otimes n} \equiv \underbrace{H \otimes H \otimes \ldots \otimes H}_{n \text{ times}} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle\langle y|,$$

where the sum is over all n-bit binary strings $x, y \in \{0,1\}^n$, and,

$$x \cdot y = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots (\vee x_n \wedge y_n).$$

Here, $\vee$ is the bitwise binary OR operation and $\wedge$ is the bitwise AND operation.

(iv) Given the observables

$$P = \sigma_Z, Q = \sigma_X, R = \frac{-\sigma_Z - \sigma_X}{\sqrt{2}}, S = \frac{\sigma_Z - \sigma_X}{\sqrt{2}}. \tag{1}$$

Evaluate the expectation values $\langle Q \otimes R \rangle_{|\beta_{00}\rangle}$, $\langle P \otimes S \rangle_{|\beta_{00}\rangle}$, in the state

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

(2) The Uncertainty Principle **[4 Marks]**

The commutator $[A, B]$ and anti-commutator $\{A, B\}$ for a pair of linear operator A, B acting on a Hilbert space $\mathcal{H}$ are defined as follows:

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA.$$

(i) Verify that

$$AB = \frac{[A, B] + \{A, B\}}{2}. \tag{2}$$

(ii) Prove the *Robertson-Schrödinger* formulation of the uncertainty principle: the product of the standard-deviations of the observed values of A, B, when measured *independently* on *identically prepared* copies of state $|\psi\rangle$ satisfies,

$$\left(\Delta_{|\psi\rangle} A\right) \left(\Delta_{|\psi\rangle} B\right) \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|. \tag{3}$$

In other words, the standard-deviations corresponding to measurements of non-commuting (*incompatible*) observables, on identically prepared states of a system, cannot both be made arbitrarily small.
Hint: Use the *Cauchy-Schwarz inequality*, which states that any two vectors $|v\rangle$, $|w\rangle$ in a Hilbert space $\mathcal{H}$ satisfy,

$$|\langle v|w\rangle|^2 \leq \langle v|v\rangle \langle w|w\rangle.$$

(iii) Consider a single qubit prepared in an eigenstate of σ_Y corresponding to the $+1$ eigenvalue. Verify the uncertainty principle for the observables σ_X and σ_Z measured on such a system, by explicitly evaluating the LHS and RHS of Eq. (3).

(3) The trace function **[3 Marks]**

The trace of an operator A is defined as the sum of its diagonal elements. That is, if $A = \sum_{ij} a_{ij} |i\rangle\langle j|$ in some orthonormal basis $\{|i\rangle\}$, then,

$$\text{Tr}[A] = \sum_i a_{ii}.$$

(i) Show that the trace is cyclic and linear: $\text{Tr}[AB] = \text{Tr}[BA]$, $\text{Tr}[A + B] = \text{Tr}[A] + \text{Tr}[B]$.

(ii) Show that for any pure state $|\psi\rangle$, $\mathrm{Tr}[A|\psi\rangle\langle\psi|] = \langle\psi|A|\psi\rangle$.

(iii) Show that the trace is invariant under a unitary transformation of the form $A \to UAU^\dagger$, where U is a unitary matrix:

$$\mathrm{Tr}[A] = \mathrm{Tr}[UAU^\dagger].$$

(4) All about Pauli operators [5 Marks]

Recall that the 2×2 Pauli operators (expressed in the eigenbasis of σ_Z: $\{|0\rangle, |1\rangle\}$) are,

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Let $\vec{v} = (v_X, v_Y, v_Z)$ be any real, three-dimensional unit vector and θ a real number. Show that

$$e^{i\theta\vec{v}.\vec{\sigma}} = \cos\theta\, I + i\sin\theta\,(\vec{v}.\vec{\sigma}),$$

where, $\vec{\sigma} = \{\sigma_X, \sigma_Y, \sigma_Z\}$.

(ii) More generally, let $f(.)$ be a function from complex numbers to complex numbers. Then, show that

$$f(\theta\vec{v}.\vec{\sigma}) = \left(\frac{f(\theta) + f(-\theta)}{2}\right) I + \frac{f(\theta) - f(\theta)}{2}\, \vec{v}.\vec{\sigma}.$$

(iii) Show that $\vec{v}.\vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by

$$P_\pm = \frac{1}{2}\left(\mathbb{I} + \vec{v}.\vec{\sigma}\right).$$

(iv) Show that the operators $\{I, \sigma_X, \sigma_Y, \sigma_Z\}$ are mutually orthogonal in terms of the *Hilbert-Schmidt inner-product* which is defined as follows for a pair of linear operators A, B :

$$\langle A, B\rangle = \mathrm{tr}[A^\dagger B].$$

What is the normalization factor required so as to make this set an orthonormal basis for the space $\mathbb{M}_2(\mathbb{C})$ of 2×2 complex matrices?

(5) Eavesdropping and disturbance **[5 Marks]**

Anita wants to send a classical message to Bharat by making use of a pair of quantum states $\{|u\rangle, |v\rangle\}$, which she is able to prepare in her lab. The states $|u\rangle$, $|v\rangle$ can be expressed in a suitable basis as,

$$|u\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \; |v\rangle = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}, \; 0 \le \alpha \le \frac{\pi}{4}. \tag{4}$$

Anita decides at random to send either $|u\rangle$ or $|v\rangle$ to Bharat, who then has to make a measurement to decide which state she sent. Since the two states are not orthogonal ($|\langle u|v\rangle| = \sin(2\alpha)$), he cannot distinguish the states perfectly. Bharat therefore settles for a procedure which is successful only some of the time; he performs a three-outcome POVM defined by the operators:

$$\begin{aligned} E_{\bar{u}} &= A(I - |u\rangle\langle u|), E_{\bar{v}} = A(I - |v\rangle\langle v|), \\ E_{\phi} &= (1 - 2A)I + A(|u\rangle\langle u| + |v\rangle\langle v|), \end{aligned} \tag{5}$$

where A is positive real number. If he obtains outcome $\bar{u}$, Bharat knows for sure that the state $|v\rangle$ was sent; if he obtains outcome $\bar{v}$, he knows with certainty that the state $|u\rangle$ was sent; if however, he obtains outcome ϕ, then his measurement is inconclusive. It is easy to check that

$$E_{\bar{u}} + E_{\bar{v}} + E_{\phi} = I,$$

so that this set of three operators constitutes a valid POVM.

(i) How should Bharat choose the value of A so as to minimize the probability of getting outcome ϕ? What is the minimal probability of obtaining outcome ϕ, assuming that Anita sent states $|u\rangle$, $|v\rangle$ with equal probability? [**Hint:** If A is too large, E_{ϕ} will have negative eigenvalues, and the operators in Eq. (5) will no longer constitute a POVM.]

(ii) Charlie, who is eavesdropping on the conversation, would also like to find out which state Anita is sending Bharat. He intercepts each qubit that Anita sends, by performing an orthogonal measurement that projects onto the $\{|0\rangle, |1\rangle\}$ basis, where,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If he obtains outcome 0 he sends the state $|u\rangle$ to Bharat, and on obtaining outcome 1 he sends $|v\rangle$. Thus, every time Bharat obtains a "conclusive" outcome ($\bar{u}$ or $\bar{v}$), Charlie knows with certainty which state he has.

However, Charlie's tampering does cause "detectable" errors; sometimes Bharat obtains a conclusive outcome that differs from what Anita sent. What is the probability of such an error, when Bharat obtains a conclusive outcome?

(6) Positive operators **[3 Marks]**

An operator A on a linear vector space V is said to be a *positive* operators if for every vector $|v\rangle \in V$, the inner product $(|v\rangle, A|v\rangle)$ is a real, non-negative number. That is,

$$(|v\rangle, A|v\rangle) = \langle v|A|v\rangle \geq 0.$$

(i) Show that the eigenvalues of a positive operator are all non-negative.

(ii) Show that a positive operator is necessarily Hermitian. [**Hint:** Show that an arbitrary operator A can be written as $B + iC$, where B and C are Hermitian.]

(iii) Show that for any operator M, $M^\dagger M$ is positive.