Department of Physics
Indian Institute of Technology, Madras
PH5840 Quantum Computation and Quantum Information

Assignment 2 4 September 2017 (Due: 14 September 2017) 25 Marks

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(1) Unitary freedom in the ensemble for density matrices [3 Marks]
Consider the following qubit density operator:

$$\rho = \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{2}|+\rangle\langle +|,$$

where the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
Obtain the unitary matrix that relates the ensemble $\{\frac{1}{4}|0\rangle, \frac{1}{4}|1\rangle, \frac{1}{2}|+\rangle\}$ to the "diagonal" ensemble $\{\lambda_1|e_1\rangle, \lambda_2|e_2\rangle\}$, where $\{|e_1\rangle, |e_2\rangle\}$ are the (normalized) eigenstates of ρ and $\{\lambda_1, \lambda_2\}$ the corresponding eigenvalues.

(2) Probability distributions consistent with a mixed state [6 Marks]
Consider a density operator ρ with the spectral decomposition

$$\rho = \sum_k \lambda_k |e_k\rangle\langle e_k|. \tag{1}$$

We would like to realize this density operator as an ensemble of pure states $\{|\phi_i\rangle\}$, where each state $|\phi_i\rangle$ is picked with a probability p_i. In particular, given a probability distribution $\{p_i\}$, we want to know if there exist pure states $\{|\phi_i\rangle\}$ such that

$$\rho = \sum_i p_i |\phi_i|\rangle\langle\phi_i|. \tag{2}$$

Show that Eq. (2) holds if and only if the probability distributions $p \equiv \{p_i\}$ and $\lambda \equiv \{\lambda_k\}$ are related as follows:

$$p_i = \sum_k D_{ik}\lambda_k, \tag{3}$$

where, D is a *doubly stochastic matrix*, that is, the elements of D are non-negative and satisfy $\sum_i D_{ik} = \sum_k D_{ik} = 1$. (The fact that the columns sum to one ensures that the matrix D maps probability distributions to probability distributions. The fact that the rows also sum to one ensure that D maps the uniform distribution to itself.)

Note: When two probability distributions q and λ are related by a double stochastic matrix (as in Eq. (3)), we say that q is *majorized* by λ, denoted as $q \prec \lambda$. The aim of this exercise is to show that the density operator ρ can be decomposed in terms of a probability distribution $p \equiv \{p_i\}$ if and only if p is majorized by the eigenvalue vector $\lambda \equiv \{\lambda_k\}$.

Hint: Use the unitary freedom in the ensemble associated with a density matrix (*Schrödinger-Hughston-Jozsa-Wootters Theorem*(. You may also find *Horn's Lemma* useful: If $p \prec q$, there exists a real unitary (orthogonal) matrix U such that $p_i = \sum_k D_{ik} q_k$, where $D_{ik} = |U_{ik}|^2$.

(3) Superdense Coding [4 (2 + 2) Marks]

Superdense Coding is a two party protocol by which one party Anita is able to communicate one of *four* classical bit strings (00, 01, 10 or 11) by simply sending across one qubit to the other party Bharat (who might be located far away from her), provided they initially share a pair of qubits in an entangled state. Specifically, suppose they initially share the Bell state,

$$|\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle.$$

Then, Anita applies one of four unitary operations on her qubit depending on the string she wants to communicate, and then sends her qubit across to Bharat. By performing a suitable measurement on both the qubits in his possession, Bharat can determine as to which classical string was sent by Anita.

(i) What are the four states resulting from the action of the unitaries I, σ_X, σ_Z and $i\sigma_Y$ on Anita's half of the Bell state? Show that these four states constitute an orthonormal basis for the space of two qubits, and hence can be distinguished by an appropriate quantum measurement.

(ii) In the super-dense coding protocol between Anita and Bharat, suppose E_A is any positive operator acting on Anita's qubit. Show that $\langle\psi|E_A \otimes I_B|\psi\rangle$ takes on the same value when $|\psi\rangle$ is any of the four Bell states. Suppose some malicious eavesdropper were to intercept Anita's qubit on the way. Can he/she infer anything about which of the four classical strings 00, 01, 10 or 11 Anita is sending to Bharat? If so, how? Or if not, why not?

(4) Schmidt Decomposition [7 (2+ 3 + 2) Marks]

(i) A simple way to arrive at the Schmidt form of a bipartite pure state $|\Psi\rangle_{AB}$ is as follows. Let $\{|i\rangle_A\} \in \mathcal{H}_A$ be the basis in which the reduced density operator $\rho_A \equiv \mathrm{Tr}_B[|\Psi\rangle\langle\Psi|]$ is diagonal, and let $\{\lambda_i\}$ be its eigenvalues. Let $|j\rangle_B$ be any orthonormal basis in $\mathcal{H}_B$. Suppose the state $|\Psi\rangle$ when expanded in the basis $\{|i\rangle_A \otimes |j\rangle_B\}$, has the following form:

$$|\Psi\rangle = \sum_{i,j} c_{ij}|i\rangle_A|j\rangle_B.$$

Show that the states

$$|\tilde{i}\rangle_B = \sum_j c_{ij} |j\rangle_B$$

are mutually orthogonal and have norm $\| \, |\tilde{i}\rangle_B \, \| = \sqrt{\lambda_i}$. Hence, show that the set $\{|i\rangle_A\}$ and the normalized set $\{|e_i\rangle_B = |\tilde{i}\rangle_B / \, \| \, |\tilde{i}\rangle_B \, \|\}$ constitute Schmidt bases for the state $|\Psi\rangle$ by explicitly writing down its Schmidt form.

(ii) Obtain the Schmidt form of the following states:

$$\begin{aligned} |\Psi_1\rangle_{AB} &= \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}, \\ |\Psi_2\rangle_{AB} &= \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \\ |\Psi_3\rangle_{AB} &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \end{aligned}$$

For each of the states $|\Psi_i\rangle$, evaluate the reduced state $\rho_A^{(i)} \equiv \mathrm{Tr}_B[\, |\Psi_i\rangle\langle\Psi_i| \,]$ on subsystem $\mathcal{H}_A$ and write down its eigenvalues.

(iii) Recall that the Shannon entropy H of a probability distribution $\{p_i\}$, is defined as follows:

$$H(\,\{p_i\}\,) \equiv -\sum_i p_i \log p_i.$$

Calculate the Shannon entropy of the eigenvalue distributions of each of the reduced states obtained above. Hence determine as to which is the most entangled state and which is least entangled.

(5) Tsirelson's bound **[5 Marks]**

Let $P = \vec{p}.\vec{\sigma}$, $Q = \vec{q}.\vec{\sigma}$, $R = \vec{r}.\vec{\sigma}$ and $S = \vec{s}.\vec{\sigma}$, where $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ are real unit vectors in three dimensions.

(i) Show that

$$(P \otimes R + Q \otimes R + Q \otimes S - P \otimes S)^2 = 4I + [P, Q] \otimes [R, S]\,.$$

(ii) Use the above result to show that

$$\langle P \otimes R\rangle + \langle Q \otimes R\rangle + \langle Q \otimes S\rangle - \langle P \otimes S\rangle \leq 2\sqrt{2}.$$

In other words, this is the maximum possible violation of the Bell-CHSH inequality allowed by quantum mechanics.

(iii) Show that this upper bound is attained for the choice of observables

$$P = \sigma_Z, Q = \sigma_X, R = \frac{-\sigma_Z - \sigma_X}{\sqrt{2}}, S = \frac{\sigma_Z - \sigma_X}{\sqrt{2}},$$

in the state

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$