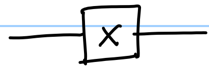
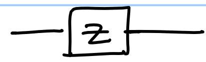
PH 5840 Lecture - 18,19

Note Title 9/24/2015

Note: (a) In a quantum circuit, Paulis are represented as:

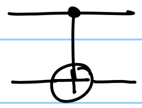
σ_x : "X-gate" 

σ_z : "Z-gate" 

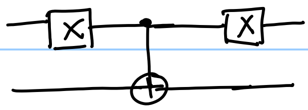
* Measurement in the quantum circuit model:-

(i) **Deferred Measurement:-** Measurements can always be moved to the end of a circuit by use of controlled operations.

Using CNOT:



(or)



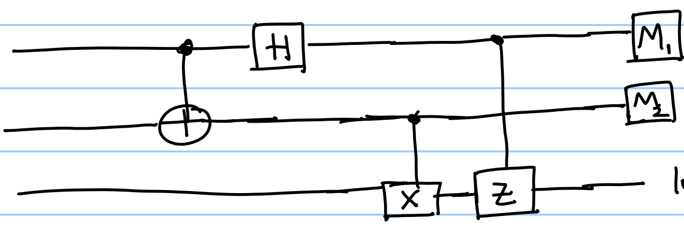
Flip target if control is set to '1'

Flips target if control is set to zero!

Eg. Teleportation:-

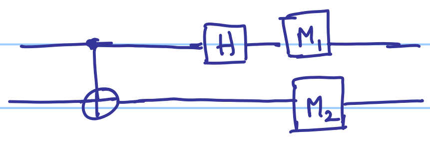
$|\psi\rangle$

$|\beta_{00}\rangle$



$|\psi\rangle$ (unknown state)

(ii) Measuring in the Bell basis :-



Measurement in $\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$

* Universal Quantum Gates

Defn: A set of gates $\mathcal{G} = \{G_1, G_2, \ldots G_N\}$ is universal for quantum computation if <u>any unitary operation</u> can be <u>approximated</u> to <u>arbitrary accuracy</u> by a quantum circuit using only those gates.

One set of universal gates: $\{H, S, \text{CNOT}, G_{\pi/8}\}$.

* Two-step proof:-

(1) Arbitrary U can be <u>exactly</u> realized as a product of single-qubit unitaries and CNOT

(2) Any single-qubit unitary can be <u>approximated</u> to arbitrary accuracy using $H, S, G_{\pi/8}$.

(1) Exact Universality:-

Step (1a): Arbitrary $d \times d$ unitary $\rightarrow$ 2-level unitary

(acts nontrivially on some 2-dim subspace of d-dim space)

Eg. Consider the 3-qubit gate:-

$$U = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 1 & & & & & & 0 \\ 0 & & 1 & & 0 & & & 0 \\ 0 & & & 1 & & & & 0 \\ 0 & & & & 1 & & & 0 \\ 0 & & 0 & & & 1 & & 0 \\ 0 & & & & & & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & d \end{pmatrix}$$

Acts non-trivially only on span of $\{|000\rangle, |111\rangle\}$

We show that :- Say U is a $d \times d$ unitary,

$\exists$ 2-level unitaries $U_1, U_2, U_3 \ldots$ etc

such that $U_k U_{k-1} U_{k-2} \cdots U_3 U_2 U_1 U = I$

$\therefore U = U_1^\dagger U_2^\dagger U_3^\dagger \cdots U_k^\dagger$.

Example:- $U = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ (3×3)

(i) If $d = 0$, let $U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

otherwise, $U_1 = \begin{pmatrix} \frac{a^*}{\sqrt{|a|^2+|d|^2}} & \frac{d^*}{\sqrt{|a|^2+|d|^2}} & 0 \\ \frac{d}{\sqrt{|a|^2+|d|^2}} & \frac{-a}{\sqrt{|a|^2+|d|^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\left(\quad \right) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$

$U_1 U = \begin{pmatrix} a' & b' & c' \\ 0 & e' & f' \\ g & h & j \end{pmatrix}$

(ii) Illly if $g = 0$, $U_2 = \begin{pmatrix} a'^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, otherwise,

$U_2 = \begin{pmatrix} \frac{a'^*}{\sqrt{|a'|^2+|g|^2}} & 0 & \frac{g^*}{\sqrt{|a'|^2+|g|^2}} \\ 0 & 1 & 0 \\ \frac{a}{\sqrt{}} & & \frac{-g}{\sqrt{}} \end{pmatrix}$

$$\therefore \quad U_2 U_1 U = \begin{pmatrix} 1 & b'' & c'' \\ 0 & e'' & f'' \\ 0 & h'' & j'' \end{pmatrix} = V$$

Since V is unitary, $b'' = c'' = 0$

$$\therefore \quad U_2 U_1 U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e'' & f'' \\ 0 & h'' & j'' \end{pmatrix} = U_3^{\dagger}$$

$$\therefore \quad U = U_1^{\dagger} U_2^{\dagger} U_3^{\dagger}$$

* Generally, for a $d \times d$ unitary U_d,

using $(d-1)$ two-level matrices $\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \boxed{U_{d-1}} \end{pmatrix}$

$U_{d-1} \rightarrow (d-2)$ two-level matrices $\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \boxed{U_{d-2}} \end{pmatrix}$

etc.

$\therefore \quad U_d = V_1 V_2 \dots V_k$ (two-level unitaries)

with $k \leq (d-1) + (d-2) + \dots 1$

$$= \frac{d(d-1)}{2}.$$

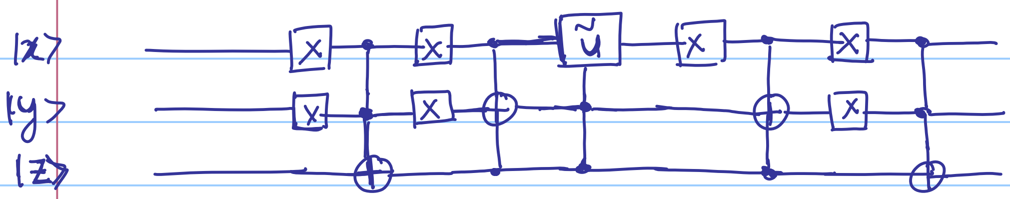
Step (1b):- Any two-level unitary can be realized exactly using single-qubit unitaries and CNOT.

Exact universality: $\mathcal{G} = \{$ single-qubit unitary $SU(2)$, CNOT $\}$

Example:

$$U = \begin{pmatrix} a & & & 0 & & & & b \\ & 1 & & & & & & \\ & & 1 & & & & 0 & \\ & 0 & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ c & & & & & & & d \end{pmatrix}_{8\times 8}$$

$$\tilde{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acting on } \{|000\rangle, |111\rangle\}$$



$000 \longrightarrow 001 \longrightarrow 011$

* Step (2) :- Approximate universality :

- Quantifying error :

When V is implemented instead of U : $e(U,V)$

$$e(U,V) = \max_{|\psi\rangle} \| (U-V)|\psi\rangle \|,$$

- Recall, $\| |\phi\rangle \| = \sqrt{\langle\phi|\phi\rangle}$.
- If $e(U,V)$ is small, any measurement on $V|\psi\rangle$ gives approximately the same statistics as a measurement on $U|\psi\rangle$.

Let M_i be an arbitrary element of a POVM.

$p_U(i) \equiv$ prob. of outcome 'i' for the state $U|\psi\rangle$

$p_V(i) \equiv$ " for state $V|\psi\rangle$.

Then, $|p_U(i) - p_V(i)| \leq 2\, e(U,V)$.

Proof: $\left|p_U(i) - p_V(i)\right| = \left|\langle\psi|U^\dagger M_i U|\psi\rangle - \langle\psi|V^\dagger M_i V|\psi\rangle\right|$

Let $|\xi\rangle = (U - V)|\psi\rangle$.

Then, $\left|p_U(i) - p_V(i)\right| = \left|\langle\psi|U^\dagger M|\xi\rangle + \langle\xi|MV|\psi\rangle\right|$

(△le inequality) $\leq |\langle\psi|U^\dagger M|\xi\rangle| + |\langle\xi|MV|\psi\rangle|$

(Cauchy-Schwarz) $\leq \| U|\psi\rangle\| \, \||\xi\rangle\| + \||\xi\rangle\| \, \|V|\psi\rangle\|$

$= 2\|(U-V)|\psi\rangle\| = 2E(U,V)$ //.

- Further, the errors add up <u>linearly</u> for a sequence of unitaries.