PH 5840 Lecture - 2

Note Title 8/5/2015

① Computing with qubits :-

(a) Deutsch algorithm (see Lecture - 1)

(b) Quantum copying circuit ?

No - cloning principle : (Wootters and Zurek, Nature 299, 802 1982)

Is it possible to make a copy of an unknown quantum state ? No!

$$\underset{\text{data}}{|\psi\rangle} \otimes \underset{\text{"target" slot}}{|s\rangle} \xrightarrow{U} |\psi\rangle \otimes |\psi\rangle$$

Suppose it works for $|\psi\rangle, |\phi\rangle$.

$$|\psi\rangle|s\rangle \xrightarrow{U} |\psi\rangle|\psi\rangle$$

$$|\phi\rangle|s\rangle \xrightarrow{U} |\phi\rangle|\phi\rangle$$

$$\Rightarrow \langle\phi|\psi\rangle = (\langle\phi|\psi\rangle)^2 \Rightarrow \langle\phi|\psi\rangle = 0 \text{ or } \langle\phi|\psi\rangle = 1$$

* Works only for orthogonal states!

i.e. No universal quantum copying machine is possible!

* Resolution : Use "entanglement" and implement "Quantum teleportation"!

② **Fundamentals**: Quantum states, Operators & Measurements.

(2a) Quantum states are associated with unit-vectors in a Hilbert space.

- $|\psi\rangle \in \mathcal{H}^d$ (d-dimensional Hilbert space) $(\langle\psi|\psi\rangle = 1)$

- Hilbert space: linear vector space equipped with an inner-product, which is complete (every Cauchy sequence in the space converges w.r.t. the norm defined by the inner product)

* Finite-dimensions:- Inner-product space $\Updownarrow$ Hilbert space.

Egs. (i) $\mathbb{C}^d$ (d-dim complex vector space)

$$|v\rangle \in \mathbb{C}^d \Leftrightarrow \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$$ d-tuple with $v_i \in \mathbb{C}$

- For $|v\rangle, |w\rangle \in \mathbb{C}^d$

Inner-product: $$\langle w|v\rangle = (w_1^* \; w_2^* \; \cdots \; w_d^*) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$$

$$= \sum_i w_i^* v_i .$$

Norm: $$\| |v\rangle \| = \sqrt{\langle v|v\rangle} = \sum_i |v_i|^2$$

(ii) $M_{d\times d}(\mathbb{C})$: Space of $d\times d$ complex matrices.

$$M_{2\times 2}(\mathbb{C}) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad m_{ij} \in \mathbb{C}$$

Inner-product: $\langle A|B\rangle = Tr(A^\dagger B)$
(Hilbert-Schmidt inner product)

Norm: $\|A\| = \sqrt{\mathrm{Tr}(A^\dagger A)}$

Exercise: Show that $\{I, \sigma_x, \sigma_y, \sigma_z\}$ forms an ON basis for $M_{2\times 2}$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left(\langle \sigma_i | \sigma_j \rangle = 2\delta_{ij}\right)$$

(iii) Space of sq-integrable functions : $\psi(x) : [0,1] \to \mathbb{C}$

($\mathcal{L}^2$ space) $\qquad \int_0^1 |\psi(x)|^2 dx < \infty$

(2b) Linear Operators: $|v\rangle, |w\rangle \in \mathcal{H}$

$\mathcal{L}(\mathcal{H}) \ni |v\rangle\langle w| : \mathcal{H} \to \mathcal{H}$

$(|v\rangle\langle w|)(|u\rangle) = \langle w|u\rangle |v\rangle$

$|v\rangle\langle v| = P_{|v\rangle}$ projection $P^2 = P$

* Matrix representation: $\{|e_i\rangle\}$ is an ON basis for $\mathcal{H}$

$$A|e_i\rangle = |u_i\rangle = \sum_j a_{ij} |e_j\rangle$$

$$\langle e_j | A | e_i \rangle = a_{ij}$$

* Outer-product representation:

completeness relation: $\sum_i |e_i\rangle\langle e_i| = I$

$$\Rightarrow A = IAI = \sum_i |e_i\rangle\langle e_i| A |e_j\rangle\langle e_j|$$

$$A = \sum_{ij} a_{ij} |e_i\rangle\langle e_j|$$

* Diagonal representation:

A is "diagonalizable" if $\exists \{|f_i\rangle\}$ ON basis such that $A = \sum_i \lambda_i |f_i\rangle\langle f_i|$

λ_i ↑ eigenvalues; $|f_i\rangle\langle f_i|$ ↳ eigenvectors

More generally, $A = \sum_i x_i P_i$ ↳ eigen projectors

When is A diagonalizable?

* $\underline{\text{Spectral Theorem}}$: Any normal operator M on $\mathcal{H}$ diagonalizable w.r.t. some ON basis in $\mathcal{H}$.

M is normal $\Leftrightarrow MM^\dagger = M^\dagger M \Leftrightarrow [M, M^\dagger] = 0$

Eg: Self-adjoint / Hermitian: $M^\dagger = M$.

Unitary: $U^\dagger U = UU^\dagger = I$

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References: Nielsen and Chuang, chapter-2

$\boxed{\text{Proof of Spectral Theorem: Box 2.2}}$