PH 5840 Lectures - 3, 4

Note Title 8/6/2015

Fundamentals: Measurement (Projection, POVM)
Uncertainty Principle, Density Operator

(1) Dynamics: Evolution of $|\psi\rangle$ is described by a unitary operator.

$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$$

(2) Measurement: Projective, POVMs

(2a) Projective Measurement M: associated with a set of projection operators $\{P_i^M\}$.
such that $\sum_i P_i = I$.

- Measuring M on $|\psi\rangle \longrightarrow \dfrac{P_i|\psi\rangle\langle\psi|P_i}{Tr(P_i|\psi\rangle\langle\psi|)}$ with prob. $p(i)$

$$p(i) = |\langle\psi|m_i\rangle|^2 = Tr(|\psi\rangle\langle\psi||m_i\rangle\langle m_i|)$$
$$= Tr(|\psi\rangle\langle\psi|P_i)$$

NOTE: Trace & the Dirac notation:-

$$Tr(|\psi\rangle\langle\psi|P_i) = \sum_i \langle e_i|(|\psi\rangle\langle\psi|P_i)|e_i\rangle$$
(where $\{|e_i\rangle\}$ is an ON basis for $\mathcal{H}$)

$$\therefore\ Tr(|\psi\rangle\langle\psi|P_i) = \sum_i \langle e_i|\psi\rangle\langle\psi|P_i|e_i\rangle$$
exchange

$$= \sum_i \langle\psi|P_i\underbrace{|e_i\rangle\langle e_i|}\psi\rangle$$

$$= \langle\psi|P_i|\psi\rangle \quad \left(\text{Since } \sum_i |e_i\rangle\langle e_i| = I\right)$$

$$= |\langle\psi|m_i\rangle|^2 \quad \text{QED!}$$

- M is self-adjoint $\Rightarrow$ $M = \sum_i m_i |m_i\rangle\langle m_i|$

$$= \sum_i m_i P_i$$

- Measurement transformation:-

$$|\psi\rangle\langle\psi| \longrightarrow \sum_i P_i |\psi\rangle\langle\psi| P_i = \rho$$

- Post-measurement state:

Ensemble $\rho \equiv \{ p(i), |m_i\rangle\langle m_i| \}$

$\therefore$ Density operator: Mixture, not a superposition!

- Expectation value: $\langle M \rangle_{|\psi\rangle} = \langle\psi|M|\psi\rangle$

$$\mathbb{E}_{|\psi\rangle}(M) = \sum_i p(i)\, m_i = \sum_i |\langle\psi|m_i\rangle|^2 m_i$$

$$= \langle\psi| \left(\sum_i m_i |m_i\rangle\langle m_i|\right) |\psi\rangle = \langle\psi|M|\psi\rangle$$

Example: Measuring σ_x on state $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

σ_x has eigenstates $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$P_+ = |+\rangle\langle +|, \quad P_- = |-\rangle\langle -|$$

$$\therefore \quad p(+) = |\langle\psi|+\rangle|^2 = \mathrm{Tr}\left[(|\psi\rangle\langle\psi|)(|+\rangle\langle+|)\right]$$

$$\left.\begin{aligned} p(+) &= \frac{|\alpha_0 + \alpha_1|^2}{2} \\ p(-) &= \frac{|\alpha_0 - \alpha_1|^2}{2} \end{aligned}\right\} \quad p(+) + p(-) = 1$$

- Post-measurement state: $\rho = p(+)\,|+\rangle\langle+| + p(-)\,|-\rangle\langle-|$

(Note: $\mathrm{Tr}(\rho) = 1$)

- Expectation value: $\langle \sigma_x \rangle_{|\psi\rangle} = \alpha_0^* \alpha_1 + \alpha_1^* \alpha_0$

$$\langle \Delta\sigma_x \rangle_{|\psi\rangle} = \langle \sigma_x^2 \rangle_{|\psi\rangle} - \langle \sigma_x \rangle_{|\psi\rangle}^2 = 1 - \langle \sigma_x \rangle_{|\psi\rangle}^2$$

(2b) POVM : Positive operator valued measure

$$M = \{M_i\}, \quad 0 < M_i < I, \quad \sum_i M_i = I$$

* prob of outcome 'i' : $p(i) = \mathrm{Tr}(|\psi\rangle\langle\psi| M_i)$

Correspondingly, $|\psi\rangle \rightarrow \sqrt{M_i}|\psi\rangle$

Post-measurement state : $\sum_i \sqrt{M_i}|\psi\rangle\langle\psi|\sqrt{M_i}$

Eg: Task: To distinguish between $|0\rangle$ and $|+\rangle$

POVM with elements: $M_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1|$

$M_2 = \frac{\sqrt{2}}{1+\sqrt{2}} (|-\rangle\langle -|)$

$M_3 = I - M_1 - M_2$

$$M_1 + M_2 = |1\rangle\langle 1| + |-\rangle\langle -| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

Has e.values $1 \pm \sqrt{2}$

$\Rightarrow M_1 + M_2 > I$!

$\Rightarrow$ Zero-error discrimination strategy:

If system is in state $|0\rangle$, outcome '1' (M_1) cannot occur!

$|-\rangle$, outcome '2' (M_2) cannot occur!

i.e. in state $|0\rangle$, $p(1) = 0$

$|-\rangle$, $p(2) = 0$

i.e. if M_1 is realized, system is in state $|-\rangle$

M_2 " , " state $|0\rangle$

M_3 " $\longrightarrow$ no information about the state of the system!

(3) Non-orthogonal quantum states cannot be distinguished perfectly.

* 2-party game: $A \longrightarrow B$

$\{|\psi_1\rangle, |\psi_2\rangle, \cdots |\psi_N\rangle\}$

A sends $|\psi_i\rangle$ $(1 \le i \le N)$; Can B identify 'i'?

(a) If $\{|\psi_i\rangle\}$ are mutually orthogonal,

B performs the measurement: $M_i = |\psi_i\rangle\langle\psi_i|$

$(1 \le i \le N)$

For completeness: $M_{N+1} = I - \sum_i |\psi_i\rangle\langle\psi_i|$

(b) If $|\psi_1\rangle, |\psi_2\rangle$ are non-orthogonal:-

* Suppose $\exists$ a measurement M with operators $\{M_1, M_2\}$

such that $\langle\psi_1|M_1|\psi_1\rangle = 1$; $\langle\psi_2|M_2|\psi_2\rangle = 1$ — ⊛

$\sum_i M_i = I$ $\Rightarrow \langle\psi_1|M_2|\psi_1\rangle = 0$

$\Rightarrow \sqrt{M_2}|\psi_1\rangle = 0$

Let $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\psi_1^\perp\rangle$; $|\alpha|^2 + |\beta|^2 = 1$

$\therefore \sqrt{M_2}|\psi_2\rangle = \beta\sqrt{M_2}|\psi_1^\perp\rangle$

$\Rightarrow \langle\psi_2|M_2|\psi_2\rangle = |\beta|^2 \langle\psi_1^\perp|M_2|\psi_1^\perp\rangle$

$\le |\beta|^2 < 1$

in contradiction with ⊛ ($\langle\psi_2|M_2|\psi_2\rangle = 1$!)

QED.

(4) Uncertainty Principle :-

Standard deviation : $(\Delta M)_\psi^2 = \langle (M - \langle M \rangle)^2 \rangle$

$$= \langle M^2 \rangle_\psi - \langle M \rangle_\psi^2$$

$$= \langle \psi | M^2 | \psi \rangle - (\langle \psi | M | \psi \rangle)^2 .$$

For any pair of observables A, B,

$$(\Delta A)_\psi \, (\Delta B)_\psi \geq \frac{| \langle \psi | [A, B] | \psi \rangle |}{2}$$

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