(2b) <u>POVM</u> : Positive operator valued measure

$$M = \{M_i\}, \quad 0 \le M_i \le I, \quad \sum_i M_i = I$$

* prob of outcome 'i' : $p(i) = \mathrm{Tr}(|\psi\rangle\langle\psi| M_i)$

Correspondingly, $|\psi\rangle \to \sqrt{M_i}\,|\psi\rangle$

Post-measurement state : $\sum_i \sqrt{M_i}\,|\psi\rangle\langle\psi|\sqrt{M_i}$

<u>Eg:</u> <u>Task</u>: To distinguish between $|0\rangle$ and $|+\rangle$

POVM with elements:

$$M_1 = \frac{\sqrt{2}}{1+\sqrt{2}}\,|1\rangle\langle 1|$$

$$M_2 = \frac{\sqrt{2}}{1+\sqrt{2}}\,(|-\rangle\langle -|)$$

$$M_3 = I - M_1 - M_2$$

$$M_1 + M_2 = |1\rangle\langle 1| + |-\rangle\langle -| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

Has e.values $1 \pm \sqrt{2}$

$\Rightarrow \quad M_1 + M_2 > I$!

$\Rightarrow$ <u>Zero-error</u> discrimination strategy:

If system is in state $|0\rangle$, outcome '1' (M_1) cannot occur!

$|-\rangle$, outcome '2' (M_2) cannot occur!

i.e. in state $|0\rangle$, $p(1) = 0$

$|-\rangle$, $p(2) = 0$

i.e. if M_1 is realized, system is in state $|-\rangle$

M_2 " , " state $|0\rangle$

M_3 " $\to$ no information about the state of the system!

(3) Non-orthogonal quantum states cannot be distinguished perfectly.

* 2-party game: $A \longrightarrow B$

$\{|\psi_1\rangle, |\psi_2\rangle, \cdots |\psi_N\rangle\}$

A sends $|\psi_i\rangle$ $(1 \le i \le N)$; Can B identify 'i'?

(a) If $\{|\psi_i\rangle\}$ are mutually orthogonal,

B performs the measurement: $M_i = |\psi_i\rangle\langle\psi_i|$ $(1 \le i \le N)$

For completeness: $M_{N+1} = I - \sum_i |\psi_i\rangle\langle\psi_i|$

(b) If $|\psi_1\rangle, |\psi_2\rangle$ are non-orthogonal:-

* Suppose $\exists$ a measurement M with operators $\{M_1, M_2\}$

such that $\langle\psi_1|M_1|\psi_1\rangle = 1$; $\langle\psi_2|M_2|\psi_2\rangle = 1$ — ⊛

$\sum_i M_i = I$ $\Rightarrow \langle\psi_1|M_2|\psi_1\rangle = 0$

$\Rightarrow \sqrt{M_2}|\psi_1\rangle = 0$

Let $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\psi_1^{\perp}\rangle$; $|\alpha|^2 + |\beta|^2 = 1$

$\therefore \sqrt{M_2}|\psi_2\rangle = \beta\sqrt{M_2}|\psi_1^{\perp}\rangle$

$\Rightarrow \langle\psi_2|M_2|\psi_2\rangle = |\beta|^2 \langle\psi_1^{\perp}|M_2|\psi_1^{\perp}\rangle$

$\le |\beta|^2 < 1$

in contradiction with ⊛ ($\langle\psi_2|M_2|\psi_2\rangle = 1$!)

QED.

(4) Uncertainty Principle:-

Standard deviation: $(\Delta M)_\psi^2 = \langle (M - \langle M \rangle)^2 \rangle$

$$= \langle M^2 \rangle_\psi - \langle M \rangle_\psi^2$$

$$= \langle \psi | M^2 | \psi \rangle - (\langle \psi | M | \psi \rangle)^2 .$$

For any pair of observables A, B,

$$(\Delta A)_\psi \, (\Delta B)_\psi \geq \frac{|\langle \psi | [A, B] | \psi \rangle|}{2}$$

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