Fundamentals: Quantum states, Operators & Measurements.

(2a) Quantum states are associated with unit-vectors in a Hilbert space.

- $|\psi\rangle \in \mathcal{H}^d$ (d-dimensional Hilbert space)
  $(\langle\psi|\psi\rangle = 1)$

- Hilbert space: linear vector space equipped with an inner-product, which is complete
  (every Cauchy sequence in the space converges w.r.t. the norm defined by the inner product)

* Finite-dimensions:-
Inner-product space
$\Updownarrow$
Hilbert space.

Egs. (i) $\mathbb{C}^d$ (d-dim complex vector space)

$$|v\rangle \in \mathbb{C}^d \Leftrightarrow \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$$

d-tuple with $v_i \in \mathbb{C}$

- For $|v\rangle, |w\rangle \in \mathbb{C}^d$

Inner-product: $$\langle w|v\rangle = (w_1^* \; w_2^* \; \cdots \; w_d^*) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$$

$$= \sum_i w_i^* v_i .$$

Norm: $$\| |v\rangle \| = \sqrt{\langle v|v\rangle} = \sum_i |v_i|^2$$

(ii) $M_{d\times d}(\mathbb{C})$: Space of $d\times d$ complex matrices.

$$M_{2\times 2}(\mathbb{C}) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \qquad m_{ij} \in \mathbb{C}$$

Inner-product: $\langle A|B\rangle = \mathrm{Tr}(A^\dagger B)$
(Hilbert-Schmidt inner product)

Norm: $\|A\| = \sqrt{\mathrm{Tr}(A^\dagger A)}$

Exercise: Show that $\{I, \sigma_x, \sigma_y, \sigma_z\}$ forms an ON basis for $M_{2\times 2}$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left(\langle \sigma_i | \sigma_j \rangle = 2\delta_{ij} \right)$$

(iii) Space of sq-integrable functions : $\psi(x) : [0,1] \to \mathbb{C}$

($\mathcal{L}^2$ space) $\quad \int_0^1 |\psi(x)|^2 dx < \infty$

(2b) Linear Operators: $|v\rangle, |w\rangle \in \mathcal{H}$

$$\mathcal{L}(\mathcal{H}) \ni |v\rangle\langle w| : \mathcal{H} \to \mathcal{H}$$

$$(|v\rangle\langle w|)(|u\rangle) = \langle w|u\rangle |v\rangle$$

$|v\rangle\langle v| = P_{|v\rangle}$ projection $P^2 = P$

* Matrix representation: $\{|e_i\rangle\}$ is an ON basis for $\mathcal{H}$

$$A|e_i\rangle = |u_i\rangle = \sum_j a_{ij} |e_j\rangle$$

$$\langle e_j| A |e_i\rangle = a_{ij}$$

* Outer-product representation:

completeness relation: $\sum_i |e_i\rangle\langle e_i| = I$

$$\Rightarrow A = IAI = \sum_i |e_i\rangle\langle e_i| A |e_j\rangle\langle e_j|$$

$$A = \sum_{ij} a_{ij} |e_i\rangle\langle e_j|$$

* Diagonal representation:

A is "diagonalizable" if $\exists \{|f_j\rangle\}$ ON basis

such that $A = \sum_i \lambda_i |f_i\rangle\langle f_i|$

$\uparrow$ eigenvalues $\quad \hookrightarrow$ eigenvectors

More generally, $A = \sum_i x_i P_i$ ↳ eigen projectors

When is A diagonalizable?

* <u>Spectral Theorem</u>: Any normal operator M on $\mathcal{H}$ diagonalizable w.r.t. some ON basis in $\mathcal{H}$.

M is normal $\Leftrightarrow MM^\dagger = M^\dagger M \Leftrightarrow [M, M^\dagger] = 0$

Eg: Self-adjoint / Hermitian: $M^\dagger = M$.

Unitary: $U^\dagger U = UU^\dagger = I$

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References: Nielsen and Chuang, chapter-2

$\boxed{\text{Proof of Spectral Theorem: Box 2.2}}$

(2c) Dynamics: Evolution of $|\psi\rangle$ is described by a unitary operator.

$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$$

(2d) Measurement: Projective, POVMs

* Projective Measurement M: associated with a set of projection operators $\{P_i^{M}\}$.
  such that $\sum_i P_i = I$.

• Measuring M on $|\psi\rangle \longrightarrow \dfrac{P_i|\psi\rangle\langle\psi|P_i}{Tr(P_i|\psi\rangle\langle\psi|)}$ with prob. $p(i)$

$$p(i) = |\langle\psi|m_i\rangle|^2 = Tr\left(|\psi\rangle\langle\psi||m_i\rangle\langle m_i|\right)$$
$$= Tr\left(|\psi\rangle\langle\psi|P_i\right)$$

NOTE: Trace & the Dirac notation:-

$$Tr\left(|\psi\rangle\langle\psi|P_i\right) = \sum_i \langle e_i|\left(|\psi\rangle\langle\psi|P_i\right)|e_i\rangle$$

(where $\{|e_i\rangle\}$ is an ON basis for $\mathcal{H}$)

$$\therefore\ Tr\left(|\psi\rangle\langle\psi|P_i\right) = \sum_i \langle e_i|\psi\rangle\langle\psi|P_i|e_i\rangle$$

exchange

$$= \sum_i \langle\psi|P_i\underbrace{|e_i\rangle\langle e_i|}\psi\rangle$$

$$= \langle\psi|P_i|\psi\rangle \quad \left(\text{Since } \sum_i |e_i\rangle\langle e_i| = I\right)$$

$$= |\langle\psi|m_i\rangle|^2 \;//$$ QED!

- M is self-adjoint $\Rightarrow$ $M = \sum_i m_i |m_i\rangle\langle m_i|$

$$= \sum_i m_i P_i$$

- <u>Measurement transformation</u>:-

$$|\psi\rangle\langle\psi| \longrightarrow \sum_i P_i |\psi\rangle\langle\psi| P_i = \rho$$

- <u>Post-measurement state</u>:

Ensemble $\rho \equiv \{ p(i), |m_i\rangle\langle m_i| \}$

∴ <u>Density operator</u>: Mixture, <u>not</u> a superposition!

- <u>Expectation value</u>: $\langle M \rangle_{|\psi\rangle} = \langle\psi| M |\psi\rangle$

$$\mathbb{E}_{|\psi\rangle}(M) = \sum_i p(i)\, m_i = \sum_i |\langle\psi|m_i\rangle|^2 m_i$$

$$= \langle\psi| \Big(\sum_i m_i |m_i\rangle\langle m_i|\Big) |\psi\rangle = \underline{\underline{\langle\psi| M |\psi\rangle}}$$

<u>Example</u>: <u>Measuring σ_x</u> on <u>state</u> $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

σ_x has eigenstates $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$P_+ = |+\rangle\langle+|, \quad P_- = |-\rangle\langle-|$$

$$\therefore \quad p(+) = |\langle\psi|+\rangle|^2 = \mathrm{Tr}\big[(|\psi\rangle\langle\psi|)(|+\rangle\langle+|)\big]$$

$$\left.\begin{aligned} p(+) &= \frac{|\alpha_0 + \alpha_1|^2}{2} \\ p(-) &= \frac{|\alpha_0 - \alpha_1|^2}{2} \end{aligned}\right\} \quad p(+) + p(-) = 1$$

- <u>Post-measurement state</u>: $\rho = p(+) |+\rangle\langle+| + p(-) |-\rangle\langle-|$

(Note: $\mathrm{Tr}(\rho) = 1$)

- <u>Expectation value</u>: $\langle \sigma_x \rangle_{|\psi\rangle} = \alpha_0^* \alpha_1 + \alpha_1^* \alpha_0$

$$\langle \Delta\sigma_x \rangle_{|\psi\rangle} = \langle \sigma_x^2 \rangle_{|\psi\rangle} - \langle \sigma_x \rangle_{|\psi\rangle}^2 = 1 - \langle \sigma_x \rangle_{|\psi\rangle}^2$$