SINGULAR VALUE DECOMPOSITION

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PROPOSITION 1: Let $A \in \mathcal{L}(\mathcal{H})$ $\ni A \geq 0$. Then the following conditions are equivalent.

i) $\exists$ $\mathcal{K}$ and $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ $\ni$ $A = T^\dagger T$
ii) $\exists$ $B = B^\dagger$ $\ni$ $A = B^2$
iii) $\langle\phi|A|\phi\rangle \geq 0$ $\forall \{|\phi\rangle\} \in \mathcal{H}$
iv) A is self-adjoint i.e. $A = A^\dagger$ and its evalues are non-negative.

Proof: (i) implies (ii) implies (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

PROPOSITION 2: Let $A \in \mathcal{L}(\mathcal{H})$, then TFAE
i) $\langle\phi|A|\phi\rangle = 0$ $\forall$ $|\phi\rangle \in \mathcal{H}$
ii) $\langle\psi|A|\phi\rangle = 0$ $\forall$ $|\psi\rangle, |\phi\rangle \in \mathcal{H}$
iii) $A = 0$

[COROLLARY: $A = A^\dagger$ iff $\langle\phi|A|\phi\rangle \in \mathbb{R}$ $\forall$ $|\phi\rangle$
As say $\langle\phi|A|\phi\rangle = u$
then $\langle\phi|A^\dagger|\phi\rangle = u^*$ but $A = A^\dagger$
$\Rightarrow$ $\langle\phi|A|\phi\rangle = u$ $\Rightarrow$ $u = u^*$ $\Rightarrow$ $u \in \mathbb{R}$.
& vice-versa!]

Proof: (ii) $\Rightarrow$ (iii) is obvious
For (i) $\Rightarrow$ (iii)
consider a vector $|u\rangle = |\phi\rangle + |\psi\rangle$
then $(\langle\phi| + \langle\psi|)A(|\phi\rangle + |\psi\rangle) = \langle u|A|u\rangle = 0$
$\Rightarrow$ $\underbrace{\langle\phi|A|\phi\rangle}_{0} + \langle\psi|A|\psi\rangle^{=0} + \langle\phi|A|\psi\rangle$
$+ \langle\psi|A|\phi\rangle = 0$
$\Rightarrow$ $\langle\psi|A|\phi\rangle + \langle\phi|A|\psi\rangle = 0.$