SVD: Let $T \in L(\mathcal{H}, \mathcal{K})$

Then $|T| = \sqrt{T^\dagger T} \geq 0$, so admits a spectral decomposition.

$$\Rightarrow |T| = \sum_{i=1}^{N} \lambda_i |\phi_i \rangle\langle \phi_i| \quad \text{where } \lambda_i = 0 \ \forall \ i > r, \quad r \leq N$$

Then

$\{|\phi_i\rangle\}_{i=1}^{r}$ is an orthonormal basis for the Range of $|T|$

$\{|\phi_i\rangle\}_{i=1}^{N}$ is ON basis for $\mathcal{H}$.

Define: $\{|\psi_i\rangle\}_{i=1}^{r} \ni |\psi_i\rangle = \lambda_i^{-1} T |\phi_i\rangle$

& $\langle \psi_i|\psi_j\rangle = \delta_{ij}$

As $T = U|T|$

$\Rightarrow |\psi_i\rangle = \lambda_i^{-1} U |T| |\phi_i\rangle = \lambda_i^{-1} \lambda_i U |\phi_i\rangle$

llly $\langle \psi_j| = \langle \phi_j | U^\dagger \Rightarrow \langle \psi_i|\psi_j\rangle = \delta_{ij}$

then

$$\lambda_i |\psi_i\rangle = T|\phi_i\rangle$$

$$\lambda_i |\psi_i\rangle\langle\phi_i| = T|\phi_i\rangle\langle\phi_i|$$

$$\Rightarrow \sum_{i=1}^{r} \lambda_i |\psi_i\rangle\langle\phi_i| = T \sum_{i=1} |\phi_i\rangle\langle\phi_i| = T \mathbb{1}$$

$$\Rightarrow T = \sum_{i=1}^{r} \lambda_i |\psi_i\rangle\langle\phi_i|$$: singular value decomposition of T

$\{\lambda_i\}$ are the singular values of T = Eigenvalues of $|T|$

non-zero $\{\lambda_i\}$ = Rank of T.