Also, from polar decomposition, we know that T and $\sqrt{T^\dagger T}$ are related via a unitary ie $T = U\sqrt{T^\dagger T} = U|T|$

As $|T| \geq 0$

$\Rightarrow |T| = WDW^\dagger$ (can be diagnolized)

$\Rightarrow T = \underbrace{UW}_{V} DW^\dagger = VDW^\dagger$

Also,

$$T = U|T| = U\sum_{i=1}^{r} \lambda_i |\phi_i\rangle\langle\phi_i|$$

$$= \sum_{i=1}^{r} \lambda_i \, U|\phi_i\rangle\langle\phi_i|$$

Let $U|\phi_i\rangle = |\psi_i\rangle \quad \forall \; i = 1 \ldots r$

$$\Rightarrow T = \sum_{i=1}^{r} \lambda_i |\psi_i\rangle\langle\phi_i|$$

Note: we could have used $T = \sqrt{TT^\dagger}\, U$

Again $\sqrt{TT^\dagger} \geq 0$

then, $\sqrt{TT^\dagger} = \sum_{i=1}^{r} \lambda_i |e_i\rangle\langle e_i|$

Then $T = \sum_{i=1}^{r} \lambda_i |e_i\rangle\langle e_i| U$

Define

$$U^\dagger|e_i\rangle = |f_i\rangle$$

then

$$T = \sum_{i=1}^{r} \lambda_i |e_i\rangle\langle f_i|$$