Q3

Here, we need to find the Green's function:

$$\frac{d^2 G(\theta,\theta')}{d\theta^2} = -\delta(\theta-\theta') . \qquad 0<\theta<\pi ,$$

which vanishes at $\theta = 0$ & π.

For $\theta<\theta'$, $\left.\begin{array}{l}\frac{d^2 f}{d\theta^2}=0 \\ f(0)=0\end{array}\right\} \Rightarrow f_1(\theta) = A_1\theta$

For $\theta>\theta'$, $\left.\begin{array}{l}\frac{d^2 f}{d\theta^2}=0 \\ f(\pi)=0\end{array}\right\} \Rightarrow f_2(\theta) = A_2\left(1-\frac{\theta}{\pi}\right)$.

$G(\theta,\theta') = G(\theta',\theta)$

$$\Rightarrow G(\theta,\theta') = f_1(\theta) f_2(\theta') \underbrace{\Theta(\theta'-\theta)}_{\text{step function}} + f_2(\theta) f_1(\theta') \Theta(\theta-\theta') .$$

$$= A_1A_2\left(\theta\left(1-\frac{\theta'}{\pi}\right)\Theta(\theta'-\theta) + \theta'\left(1-\frac{\theta}{\pi}\right)\Theta(\theta-\theta')\right).$$

To find the value of A_1A_2, use the condition:

$$\left(f_1 f_2' - f_2 f_1'\right) = -1 .$$

$$A_1A_2\left(-\frac{\theta}{\pi} - \left(1-\frac{\theta}{\pi}\right)\right) = -1 . \quad \Rightarrow \quad A_1A_2 = -1 .$$

$$\Rightarrow \quad G(\theta,\theta') = \begin{cases} \frac{\theta\theta'}{\pi} - \theta & \text{for } \theta<\theta' \\ \frac{\theta\theta'}{\pi} - \theta' & \text{for } \theta>\theta' . \end{cases}$$

There is a discontinuity in $\frac{dG}{d\theta}$ at $\theta=\theta'$.

$$\frac{dG}{d\theta} = \begin{cases} \theta'/\pi - 1 & \text{for } \theta<\theta' \\ \theta'/\pi & \text{for } \theta>\theta' . \end{cases}$$

$$\left.\frac{dG}{d\theta}\right|_{\theta=\theta'_+} - \left.\frac{dG}{d\theta}\right|_{\theta=\theta'_-} = -1 .$$

(Q4) Green function for a 2D rectangular region $0 \le x \le a,\ 0 \le y \le b$.

$$\nabla^2 G(\vec{r}, \vec{r}\,') = -4\pi\, \delta(\vec{r} - \vec{r}\,').$$

$$= -4\pi\, \delta(x - x')\, \delta(y - y').$$

consider the following representation of δ-function in $[0, a]$:

$$\delta(x - x') = \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right).$$

Set $$G(\vec{r}; \vec{r}\,') = \frac{2}{a} \sum_{n=1}^{\infty} g_n(y, y') \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right).$$

$$\Rightarrow \quad \nabla^2 G(\vec{r}, \vec{r}\,') = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(\vec{r}, \vec{r}\,').$$

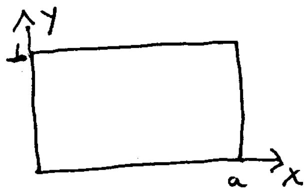
$$= \frac{2}{a} \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{a^2} + \frac{\partial^2}{\partial y^2}\right) g_n(y, y') \sin\left(\frac{n\pi x}{a}\right) \left(\frac{n\pi x'}{a}\right).$$

This must be equal to $-4\pi\, \delta(x - x')\, \delta(y - y')$.

$$= -4\pi\, \delta(y - y') \cdot \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)$$

$\therefore$ $g_n(y, y')$ satisfies:

$$\left(\frac{\partial^2}{\partial y^2} - \frac{n^2\pi^2}{a^2}\right) g_n(y, y') = -4\pi\, \delta(y - y').$$



Let $G(\vec{r}, \vec{r}\,')$ vanishes on the boundary.

$$\therefore \left.\begin{aligned} G(0, y; x', y') &= 0 \\ G(x, 0; x', y') &= 0 \\ G(a, y; x', y') &= 0 \\ G(x, b; x', y') &= 0 \end{aligned}\right\}$$

$G(0, y; x', y') = G(a, y; x', y') = 0$ is automatically satisfied for

$$G(\vec{r}; \vec{r}\,') = \frac{2}{a} \sum_{n=1}^{\infty} g_n(y, y') \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)$$

For the other two conditions, we must have

$$g(0, y') = g(b, y') = 0.$$

(b)

To find solution to the inhomogeneous equation

$$\left(\frac{\partial^2}{\partial y^2} - \frac{n^2\pi^2}{a^2}\right) g_n(y,y') = -4\pi\,\delta(y-y'), \qquad \text{——— (A)}$$

conside the solution to the homogeneous part:

$$\left(\frac{\partial^2}{\partial y^2} - \frac{n^2\pi^2}{a^2}\right) \tilde{g}_n(y,y') = 0 .$$

The most general solution is

$$\tilde{g}_n(y,y') = A_n(y')\, e^{n\pi/a\, y} + B_n(y')\, e^{-n\pi/a\, y}.$$

$g_n(y,y') = 0$ at $y=0$ and $y=b$.

A solution of the type $A_n(y') \sinh\left(\frac{n\pi}{a} y\right)$ vanishes at $y=0$, whereas solution of the type $B_n(y') \sinh\left(\frac{n\pi}{a}(b-y)\right)$ vanishes at $y=b$.

Thus solution to (A) must be of the form:

$\cancel{g_n(y,y') = A_n(y') \sinh\left(\frac{n\pi}{a} y\right) + B_n(y') \sinh\left(\frac{n\pi}{a}(b-y)\right)}$

$\cancel{\Rightarrow g_n(y',y) = A_n(y) \sinh\left(\frac{n\pi}{a} y'\right)}$

$$g_n(y,y') = A_n(y') \sinh\left(\frac{n\pi}{a} y\right) \theta(y'-y) + B_n(y') \sinh\left(\frac{n\pi}{a}(b-y)\right) \theta(y-y')$$

$$\Rightarrow g_n(y',y) = A_n(y) \sinh\left(\frac{n\pi}{a} y'\right) \theta(y-y') + B_n(y) \sinh\left(\frac{n\pi}{a}(b-y')\right) \theta(y'-y)$$

$\because$ $g_n(y,y')$ is symmetric, $g_n(y,y') = g_n(y',y)$

$$\Rightarrow \quad A_n(y') = C_n \sinh\left(\frac{n\pi}{a}(b-y')\right) \quad \text{for a constant } C_n.$$

and $B_n(y') = C_n \sinh\left(\frac{n\pi}{a} y'\right)$

Hence,

$$g_n(y,y') = C_n\Big(\sinh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y')\right) \theta(y'-y) + \sinh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y'\right) \theta(y-y')\Big)$$

To get C_n, substitute the above in (A).

C

$$\frac{\partial g_n}{\partial y} = c_n \left(\left(\frac{n\pi}{a}\right) \cosh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y')\right) \theta(y'-y) \right.$$
$$\left. - \left(\frac{n\pi}{a}\right) \cosh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y'\right) \theta(y-y') \right)$$
$$+ c_n \left(\sinh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y')\right) \left(-\delta(y'-y)\right) \right.$$
$$\left. + \sinh \frac{n\pi}{a}(b-y) \sinh\left(\frac{n\pi}{a} y'\right) \delta(y-y') \right)$$

$$= c_n\left(\frac{n\pi}{a}\right) \left(\cosh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y')\right) \theta(y'-y) - \cosh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y'\right) \theta(y-y') \right)$$
$$+ c_n \, \delta(y-y') \left(\sinh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y'\right) - \sinh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y')\right) \right)$$

$\delta(y-y') \, f(y) = \delta(y-y') \, f(y') \Rightarrow$ 2nd term is zero.

$$\therefore \frac{\partial g_n}{\partial y} = c_n\left(\frac{n\pi}{a}\right) \left[\cosh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y')\right) \theta(y'-y) - \cosh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y'\right) \theta(y-y') \right]$$

$$\therefore \frac{\partial^2 g_n}{\partial y^2} = c_n \left(\frac{n\pi}{a}\right)^2 \left[\sinh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y')\right) \theta(y'-y) + \sinh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y'\right) \theta(y-y') \right]$$
$$+ c_n\left(\frac{n\pi}{a}\right) \left[\cosh\left(\frac{n\pi}{a} y\right) \sinh \frac{n\pi}{a}(b-y') \left(-\delta(y'-y)\right) - \cosh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y'\right) \delta(y-y') \right]$$

$$= \left(\frac{n\pi}{a}\right)^2 g_n(y,y') - c_n\left(\frac{n\pi}{a}\right) \delta(y-y') \left[\cosh\left(\frac{n\pi}{a} y\right) \sinh \frac{n\pi}{a}(b-y') + \cosh \frac{n\pi}{a}(b-y) \sinh \frac{n\pi}{a} y \right]$$

$$= \left(\frac{n\pi}{a}\right)^2 g_n(y,y') - c_n\left(\frac{n\pi}{a}\right) \delta(y-y') \left[\cosh\left(\frac{n\pi}{a} y\right) \sinh\left(\frac{n\pi}{a}(b-y)\right) + \cosh\left(\frac{n\pi}{a}(b-y)\right) \sinh\left(\frac{n\pi}{a} y\right) \right]$$

$$= \left(\frac{n\pi}{a}\right)^2 g_n(y,y') - c_n\left(\frac{n\pi}{a}\right) \delta(y-y') \sinh\left(\frac{n\pi}{a} b\right).$$

$$\Rightarrow \frac{\partial^2 g_n}{\partial y^2} - \left(\frac{n\pi}{a}\right)^2 g_n(y,y') = -c_n\left(\frac{n\pi}{a}\right) \sinh\left(\frac{n\pi}{a} b\right) \delta(y-y')$$
$$= -4\pi \, \delta(y-y')$$

$$\Rightarrow \boxed{c_n = \frac{2a}{n} \frac{1}{\sinh\left(\frac{n\pi}{a} b\right)}}$$