DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5020 Electromagnetic Theory Assignment 3 13.02.2015 (due: 20.02.2015)

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1. Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

   (a) Write down the appropriate Green function $G(\mathbf{x}, \mathbf{x}')$.

   (b) If the potential on the plane $z = 0$ is specified to be $\phi = V$ inside a circle of radius a centered at the origin, and $\phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, φ, z).

   (c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

   $$\phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

   (d) Show that at large distances ($\rho^2 + z^2 >> a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

   $$\phi = \frac{Va^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + ...\right]$$

   Verify that the results of parts c and d are consistent with each other in their common range of validity.

2. Start with the series solution

   $$\phi(\rho, \varphi) = a_0 + b_0 \ \ln \ \rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\varphi + \beta_n)$$

   for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius b. Evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential inside the cylinder in the form of Poisson's integral:

   $$\phi(\rho, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} \phi(b, \varphi') \ \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\varphi' - \varphi)} d\varphi'$$

   What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?

3. (a) Construct the free-space Green function $G(x, ; x', y')$ for two-dimensional electrostatics by integrating $1/R$ with respect to $(z' - z)$ between the limits Z, where Z is taken to be very large. Show that apart from an inessential constant, the Green function can be written alternately as

$$G(x, y; x', y') = -\ln[(x - x')^2 + (y - y')^2] = -\ln[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')]$$

(b) Show explicitly by separation of variables in polar coordinates that the Green function can be expressed as a Fourier series in the azimuthal coordinate,

$$G = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{im(\varphi - \varphi')} g_m(\rho - \rho')$$

where the radial Green functions satisfy

$$\left(\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dg_m}{d\rho} \right) - \frac{m^2}{\rho^2} g_m(\rho, \rho') \right) = -\frac{4\pi}{\rho} \delta(\rho - \rho') \ .$$

(c) Complete the solution and show that the free-space Green function has the expansion

$$G(\rho, \varphi; \rho, \varphi) = -\ln({\rho_>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>} \right)^m \cos[m(\varphi - \varphi')]$$

where$\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'.

4. (a) Consider a two dimensional region bounded by the straight lines $\varphi = 0$ and $\varphi = \beta$, for some angle β. Solve the Laplace's equation in the above region, subjected to the condition $\phi(\mathbf{r}) = V_0$ on the boundary. (Here V_0 is a constant).

(b) Show that the Green function in the region described in the above problem can be expressed as:

$$G(\mathbf{r}, \mathbf{r}') = \frac{2}{\beta} \sum_{n=1}^{\infty} g_n(\rho, \rho') \sin\left(\frac{n\pi\varphi}{\beta} \right) \sin\left(\frac{n\pi\varphi'}{\beta} \right) \ ,$$

where $g_n(\rho, \rho')$ satisfies

$$\left(\frac{1}{\rho} \frac{d}{d\rho} \left(\frac{dg_n}{d\rho} \right) - \frac{n^2\pi^2}{\rho^2\beta^2} g_n(\rho, \rho') \right) = -\frac{4\pi}{\rho} \delta(\rho - \rho') \ .$$

(c) Show the the following expression for $g_n(\rho, \rho')$ is a solution to the above differential equation

$$g_n(\rho, \rho') = \frac{4\beta}{n} \left(\left(\frac{\rho}{\rho'} \right)^{n\pi/\beta} \theta(\rho' - \rho) + \left(\frac{\rho'}{\rho} \right)^{n\pi/\beta} \theta(\rho - \rho') \right) \ .$$

(d) Use the above problem to find the Green function appropriate for the Dirichlet boundary condition for the two dimensional region bounded by the lines $\varphi = 0, \varphi = \beta$ and $\rho = a$.