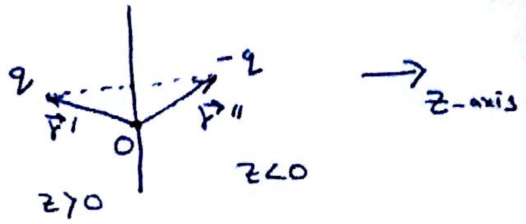
1) Use the method of images to find the Green function:

$$\vec{r} = \rho\,\hat{e}_\rho + z\,\hat{e}_z$$
$$\vec{r}\,' = \rho'\hat{e}_\rho + z'\,\hat{e}_z$$
$$\vec{r}\,'' = \rho'\hat{e}_\rho - z'\hat{e}_z\,.$$



$$\tilde{\phi}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}\,''|}\right).$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(\rho-\rho')^2+(z-z')^2}} - \frac{1}{\sqrt{(\rho-\rho')^2+(z+z')^2}}\right).$$

In $z>0$ region,

$$\nabla^2\tilde{\phi} = -\frac{q}{\epsilon_0}\delta(\vec{r}-\vec{r}\,').$$
$$\tilde{\phi}(\vec{r})\Big|_{z=0} = 0\,.$$

$$\Rightarrow \boxed{G(\vec{r},\vec{r}\,') = \cancel{\frac{q}{\epsilon_0}}\left(\frac{1}{\sqrt{(\rho-\rho')^2+(z-z')^2}} - \frac{1}{\sqrt{(\rho-\rho')^2+(z+z')^2}}\right)}.$$

If $\rho(\vec{r})$ is the charge density in $z>0$ region, ϕ_0 is the potential on the boundary ($z=0$ plane), then

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3_+}\rho(\vec{r}\,')\,G(\vec{r},\vec{r}\,')\,d^3r' - \frac{1}{4\pi}\int_S \phi_0(\vec{r}\,')\,\vec{\nabla}G\cdot d\vec{s}$$

$$\vec{\nabla}G = \left(\hat{e}_\rho\frac{\partial}{\partial\rho'} + \hat{e}_\varphi\frac{1}{\rho'}\frac{\partial}{\partial\varphi'} + \hat{e}_z\frac{\partial}{\partial z'}\right)G(\vec{r},\vec{r}\,').$$

$$= \hat{e}_\rho\left\{\frac{(\rho-\rho')}{((\rho-\rho')^2+(z-z')^2)^{3/2}} - \frac{(\rho-\rho')}{((\rho-\rho')^2+(z+z')^2)^{3/2}}\right\}$$
$$+\,\hat{e}_z\left\{\frac{(z-z')}{((\rho-\rho')^2+(z-z')^2)^{3/2}} + \frac{(z+z')}{((\rho-\rho')^2+(z+z')^2)^{3/2}}\right\}.$$

$$\vec{\nabla}G\cdot d\vec{s}\Big|_{\text{on } S} = -\frac{\partial G}{\partial z'}\Big|_{z'=0} ds = -\frac{2z}{((\rho-\rho')^2+z^2)^{3/2}}\,\rho'\,d\rho'\,d\varphi'\,.$$

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Thus the second term becomes,

$$\frac{1}{2\pi}\int \frac{z}{\left((\rho-\rho')^2+z^2\right)^{3/2}}\,\phi_0(\rho',\varphi')\;\rho'\,d\rho'\,d\varphi' .$$

If $\phi_0(\rho',\varphi')$ is independent of φ', then the above term becomes,

$$\phi_s = \int_0^{\infty} \frac{z}{\left((\rho-\rho')^2+z^2\right)^{3/2}}\,\phi_0(\rho')\;\rho'\,d\rho' .$$

consider $\phi_0(\rho') = \begin{cases} V \quad (\text{const.}) & \rho' < a \\ 0 & \rho' > a . \end{cases}$

Then, $$\phi_s(\vec{r}) = V\int_0^{a} \frac{z\,\rho'\,d\rho'}{\left((\rho-\rho')^2+z^2\right)^{3/2}} .$$

Along the axis of the circle,

$$\phi_s(\vec{r}) = V\int_0^{a} \frac{z\,\rho'\,d\rho'}{(\rho'^2+z^2)^{3/2}} . = V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)$$

At large distance, $(\rho^2+z^2 \gg a^2)$

$$\frac{1}{\left((\rho-\rho')^2+z^2\right)^{3/2}} = \frac{1}{(\rho^2+z^2)^{3/2}\left(1+\frac{\rho'^2-2\rho\rho'}{\rho^2+z^2}\right)^{3/2}}$$

$$\simeq \frac{1}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3}{2}\,\frac{\rho'^2-2\rho\rho'}{(\rho^2+z^2)}+\frac{15}{8}\,\frac{(\rho'^2-2\rho\rho')^2}{(\rho^2+z^2)^2}+\cdots\right]$$

Substituting in the integral form for $\phi_s(\vec{r})$ and carrying out the integration will give the desired result.

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2) The potential is finite at $\rho = 0$ $\Rightarrow$ $b_n = 0$ for $n = 0, 1, 2, \ldots$

$$\therefore \quad \phi(\rho, \varphi) = a_0 + \sum_{n=1}^{\infty} \rho^n \left(a_n \sin n\varphi + b_n \cos n\varphi \right).$$

The potential is a given function $\phi(b, \varphi)$ at $\rho = b$.

$$\phi(b, \varphi) = a_0 + \sum_{n=1}^{\infty} b^n \left(A_n \sin n\varphi + B_n \cos n\varphi \right).$$

$$\int_0^{2\pi} \sin n\varphi \sin m\varphi \, d\varphi = \int_0^{2\pi} \cos n\varphi \cos m\varphi \, d\varphi = \pi \delta_{mn}$$

$$\Rightarrow \quad A_n b^n \pi = \int_0^{2\pi} \phi(b, \varphi) \sin n\varphi \, d\varphi$$

$$B_n b^n \pi = \int_0^{2\pi} \phi(b, \varphi) \cos n\varphi \, d\varphi.$$

$$2\pi a_0 = \int_0^{2\pi} \phi(b, \varphi) \, d\varphi.$$

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} \phi(b, \varphi) \, d\varphi.$$

$$A_n = \frac{1}{\pi b^n} \int_0^{2\pi} \phi(b, \varphi) \sin n\varphi \, d\varphi$$

$$B_n = \frac{1}{\pi b^n} \int_0^{2\pi} \phi(b, \varphi) \cos n\varphi \, d\varphi$$

$$\Rightarrow \quad \phi(\rho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \, \phi(b, \varphi') \left\{ 1 + \sum_{n=1}^{\infty} 2\left(\frac{\rho}{b}\right)^n \left(\sin n\varphi \sin n\varphi' + \cos n\varphi \cos n\varphi' \right) \right\}$$

$$\therefore \quad \boxed{\phi(\rho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \, \phi(b, \varphi') \left\{ 1 + \sum_{n=1}^{\infty} 2\left(\frac{\rho}{b}\right)^n \cos n(\varphi - \varphi') \right\}}$$

(Q4)

Green function in plane polar coordinates:

Consider $\nabla^2 G = -4\pi\,\delta(\vec{\rho}-\vec{\rho}\,')$ in 2D.

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial G}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \varphi^2} = -4\pi\cdot\frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi').$$

Use $$\delta(\varphi-\varphi') = \frac{2}{\beta}\sum_{m=1}^{\infty}\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta}.$$

Set $$G(\vec{\rho},\vec{\rho}\,') = \frac{2}{\beta}\sum_{m=1}^{\infty} g_m(\rho,\rho')\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta}.$$

$$\Rightarrow \frac{2}{\beta}\sum_m\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m(\rho,\rho')}{d\rho}\right)\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta} - \frac{1}{\rho^2}\frac{m^2\pi^2}{\beta^2}g_m\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta}\right]$$

$$= \frac{2}{\beta}\cdot\left(-\frac{4\pi}{\rho}\right)\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta}.$$

$$\Rightarrow \boxed{\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dg_m}{d\rho} - \frac{1}{\rho^2}\frac{m^2\pi^2}{\beta^2}g_m(\rho,\rho')\right) = -\frac{4\pi}{\rho}\delta(\rho-\rho')}.$$

set $$g_m = a_m\left(\left(\rho/\rho'\right)^{m\pi/\beta}\cdot\theta(\rho'-\rho) + \left(\rho'/\rho\right)^{m\pi/\beta}\theta(\rho-\rho')\right).$$

$$\frac{dg_m}{d\rho} = a_m\left(\frac{m\pi}{\beta}\right)\cdot\frac{1}{\rho}\left(\left(\rho/\rho'\right)^{m\pi/\beta}\theta(\rho'-\rho) - \left(\rho'/\rho\right)^{m\pi/\beta}\theta(\rho-\rho')\right)$$

$$+\, a_m\left(\left(\rho/\rho'\right)^{m\pi/\beta} - \cancel{\left(\rho'/\rho\right)^{m\pi/\beta}}\right)\delta(\rho-\rho') \to 0$$

$$\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) = a_m\left(\frac{m\pi}{\beta}\right)^2\cdot\frac{1}{\rho}\left(\left(\rho/\rho'\right)^{m\pi/\beta}\theta(\rho'-\rho) + \left(\rho'/\rho\right)^{m\pi/\beta}\theta(\rho-\rho')\right)$$

$$-\, a_m\left(\frac{m\pi}{\beta}\right)\cdot\underbrace{\left(\left(\rho/\rho'\right)^{m\pi/\beta} + \left(\rho'/\rho\right)^{m\pi/\beta}\right)}_{\to 2}\delta(\rho-\rho').$$

$$= \left(\frac{m\pi}{\beta}\right)^2\cdot\frac{1}{\rho}g_m - 2a_m\cdot\left(\frac{m\pi}{\beta}\right)\delta(\rho-\rho').$$

$$\therefore\ \frac{1}{\rho}\left(\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right)\right) - \frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2 g_m(\rho,\rho') = -\frac{4\pi}{\rho}\delta(\rho-\rho')$$

$$\Rightarrow\ 2a_m\left(\frac{m\pi}{\beta}\right) = 4\pi \quad\text{or}\quad a_m = \left(\frac{2\beta}{m}\right).$$

$$\therefore\ \boxed{g_m = \frac{2\beta}{m}\left(\left(\rho/\rho'\right)^{m\pi/\beta}\theta(\rho'-\rho) + \left(\rho'/\rho\right)^{m\pi/\beta}\theta(\rho-\rho')\right)}.$$

Evaluation of the Surface Term:

Thus,

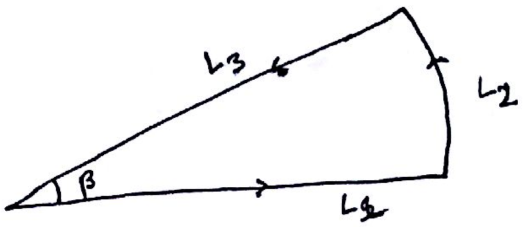
$$G(\vec{r},\vec{r}') = 4\sum_{m=1}^{\infty} \frac{1}{m}\left(\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\theta(\rho'-\rho) + \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\theta(\rho-\rho')\right) \sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta}.$$

& $\nabla^2 G = -4\pi\,\delta(\vec{r}-\vec{r}')$. $\qquad \delta(\vec{r}-\vec{r}') = \frac{1}{\rho}\delta(\rho-\rho')\,\delta(\varphi-\varphi')$.

Gauss' divergence theorem:

$$\int_S \nabla^2 G \, dS = \int_L \vec{\nabla} G\cdot\hat{n}\, dl$$

L is boundary of S.



Let $\vec{r}' = (\rho', \varphi')$, be a point inside the region.

$$\vec{\nabla} G = \frac{\partial G}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial G}{\partial \varphi}\hat{e}_\varphi.$$

$$\Rightarrow \vec{\nabla} G = 4\sum_{m=1}^{\infty}\frac{1}{m}\hat{e}_\rho\cdot\left(\frac{m\pi}{\beta}\right)\frac{1}{\rho}\left(\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\theta(\rho'-\rho) - \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\theta(\rho-\rho')\right)\sin\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta}$$

$$+\frac{4}{\rho}\sum_{m=1}^{\infty}\frac{1}{m}\hat{e}_\varphi\left(\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\theta(\rho'-\rho) + \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\theta(\rho-\rho')\right)\left(\frac{m\pi}{\beta}\right)\cdot\cos\frac{m\pi\varphi}{\beta}\sin\frac{m\pi\varphi'}{\beta}.$$

$$= \frac{4\pi}{\beta}\sum_{m=1}^{\infty}\hat{e}_\rho\frac{1}{\rho}\left(\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\theta(\rho'-\rho) - \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\theta(\rho-\rho')\right)\sin\left(\frac{m\pi\varphi}{\beta}\right)\sin\left(\frac{m\pi\varphi'}{\beta}\right)$$

$$+\frac{4\pi}{\beta}\sum_{m=1}^{\infty}\hat{e}_\varphi\frac{1}{\rho}\left(\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\theta(\rho'-\rho) + \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\theta(\rho-\rho')\right)\cos\left(\frac{m\pi\varphi}{\beta}\right)\sin\left(m\pi\varphi'/\beta\right)$$

$$\int_L \vec{\nabla} G\cdot\hat{n}\, dl = \int_{L_1}\vec{\nabla} G\cdot\hat{n}\, dl + \int_{L_2}\vec{\nabla} G\cdot\hat{n}\, dl + \int_{L_3}\vec{\nabla} G\cdot\hat{n}\, dl$$

$$\int_{L_1}\vec{\nabla} G\cdot\hat{n}\, dl = \sum_{m=1}^{\infty}(\hat{n}\cdot\hat{e}_\varphi)\int_0^{\infty}\left(\frac{4\pi}{\beta}\right)\frac{1}{\rho}\left(\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\theta(\rho'-\rho) + \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\theta(\rho-\rho')\right)\cos\left(\frac{m\pi\varphi}{\beta}\right)\Big|_{\varphi=0}\sin\left(\frac{m\pi\varphi'}{\beta}\right)$$

$$= -\frac{4\pi}{\beta}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\varphi'}{\beta}\right)\left(\int_0^{\rho'}\frac{1}{\rho}\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}d\rho + \int_{\rho'}^{\infty}\left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\frac{1}{\rho}\,d\rho\right)$$

$$= -\frac{4\pi}{\beta}\sum_{m=1}^{\infty}\left[\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\cdot\frac{1}{(m\pi/\beta)}\Bigg|_0^{\rho'} - \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\frac{1}{(m\pi/\beta)}\Bigg|_{\rho'}^{\infty}\right]\sin\left(\frac{m\pi\varphi'}{\beta}\right)$$

$$= -\frac{4\pi}{\beta} \sum_{m=1}^{\infty} \frac{\beta}{m\pi} \cdot 2 \sin\left(\frac{m\pi\varphi'}{\beta}\right)$$

$$= -\cancel{16}\,8 \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(\frac{m\pi\varphi'}{\beta}\right) .$$

$$\int_{L_2} \vec{\nabla}_\circ G \cdot \hat{n}\, dl = 0$$

$$\int_{L_3} \vec{\nabla} G \cdot \hat{n}\, dl = (\hat{n}\cdot\hat{e}_\varphi) \sum_{m=1}^{\infty} \int_{\infty}^{0} \left(\frac{4\pi}{\beta}\right) \frac{1}{\rho} \left(\left(\rho/\rho'\right)^{m\pi/\beta} \theta(\rho'-\rho) + \left(\rho'/\rho\right)^{m\pi/\beta} \theta(\rho-\rho') \right) \cos\left(\frac{m\pi\varphi}{\beta}\right)\Big|_{\varphi=\beta} \sin\left(\frac{m\pi\varphi'}{\beta}\right)$$

$$= -\sum_{m=1}^{\infty} (-1)^m \frac{8\,\cancel{16}}{m} \sin\left(\frac{m\pi\varphi'}{\beta}\right) .$$

$$\therefore \int_L \vec{\nabla} G \cdot \hat{n}\, dl = \overset{-16}{\cancel{-32}} \sum_{m=1}^{\infty} \frac{1}{(2m+1)} \sin\left(\frac{(2m+1)\pi\varphi'}{\beta}\right) .$$

Consider the step function: $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m+1)x}{(2m+1)} .$$

$$\therefore \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin[(2m+1)\pi\varphi'/\beta]}{(2m+1)} = \frac{1}{2} \quad \text{for} \quad 0 < \varphi' < \beta .$$

$$\Rightarrow \int_L \vec{\nabla} G \cdot \hat{n}\, dl = -\overset{16}{\cancel{32}} \cdot \frac{\pi}{4} = -4\pi .$$

(Q5) Green function for the region bounded by $\rho=0$ and $\varphi=\beta$ is

$$G(\vec{r},\vec{r}\,') = \frac{2}{\beta}\sum_{n=1}^{\infty} g_n(\rho,\rho')\,\sin\left(\frac{n\pi\varphi}{\beta}\right)\sin\left(\frac{n\pi\varphi'}{\beta}\right), \quad \text{where}$$

$$g_n(\rho,\rho') = \frac{2\beta}{n}\left(\left(\rho/\rho'\right)^{\frac{n\pi}{\beta}}\,\theta(\rho'-\rho) + \left(\rho'/\rho\right)^{n\pi/\beta}\,\theta(\rho-\rho')\right).$$

To this we need to add a harmonic piece

$$H(\vec{r},\vec{r}\,') = \frac{2}{\beta}\sum_{n=1}^{\infty}\tilde{g}_n(\rho,\rho')\,\sin\left(\frac{n\pi\varphi}{\beta}\right)\sin\left(\frac{n\pi\varphi'}{\beta}\right)$$

such that $\left(G(\vec{r},\vec{r}\,') + H(\vec{r},\vec{r}\,')\right)\Big|_{\rho=a} = 0$.

Since $H(\vec{r},\vec{r}\,')$ is harmonic,

$$\tilde{g}_n(\rho,\rho') = A_n(\rho')\,\rho^{n\pi/\beta} + B_n(\rho')\,\frac{1}{\rho^{n\pi/\beta}}$$

$$= a_n(\rho\rho')^{n\pi/\beta} + \frac{b_n}{(\rho\rho')^{n\pi/\beta}}.$$

$$\because\ \tilde{g}_n(\rho,\rho') = \tilde{g}_n(\rho',\rho).$$

$$\left[g_n(\rho,\rho') + \tilde{g}_n(\rho,\rho')\right]\Big|_{\rho=a} = 0.$$

for $\rho' < a$.

$$\Rightarrow \left(\frac{2\beta}{n}\right)\left(\frac{\rho'}{a}\right)^{n\pi/\beta} + a_n(a\rho')^{n\pi/\beta} + \frac{b_n}{(a\rho')^{n\pi/\beta}} = 0.$$

$$\Rightarrow b_n = 0, \qquad a_n = -\frac{2\beta}{n}\left(\frac{1}{a^2}\right)^{n\pi/\beta}.$$

$$\therefore\ \tilde{g}_n(\rho,\rho') = -\frac{2\beta}{n}\left(\frac{\rho\rho'}{a^2}\right)^{n\pi/\beta}.$$

Hence, the green function which vanishes on the boundary of the region bounded by lines $\varphi=0$, $\varphi=\beta$ & $\rho=a$

$$\boxed{G(\vec{r},\vec{r}\,') + H(\vec{r},\vec{r}\,') = \frac{2}{\beta}\sum_{n=1}^{\infty}\left(\frac{2\beta}{n}\right)\left(\left(\rho/\rho'\right)^{n\pi/\beta}\theta(\rho'-\rho) + \left(\rho'/\rho\right)^{n\pi/\beta}\theta(\rho-\rho') - \left(\frac{\rho'\rho}{a^2}\right)^{n\pi/\beta}\right)\sin\left(\frac{n\pi\varphi}{\beta}\right)\sin\left(\frac{n\pi\varphi'}{\beta}\right)}$$