Recursion Relations:

$$J_\nu(x) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{\Gamma(j+1)\,\Gamma(\nu+j+1)} \left(\frac{x}{2}\right)^{2j+\nu}.$$

Consider $\frac{d}{dx}\left(x^\nu J_\nu(x)\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\,\Gamma(\nu+j+1)} \frac{1}{2^{2j+\nu}} \frac{d}{dx}\left(x^{2j+2\nu}\right).$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j\, 2(j+\nu)}{\Gamma(j+1)\,\Gamma(\nu+j+1)} \frac{1}{2^{2j+\nu}} \cdot x^{2j+2\nu-1}.$$

$(\Gamma(n+1) = n\Gamma(n))$ $\Rightarrow$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^\nu}{\Gamma(j+1)\,\Gamma(\nu+j)} \left(\frac{x}{2}\right)^{2j+2\nu-1}$$

$$= x^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\,\Gamma(\nu+j)} \left(\frac{x}{2}\right)^{2j+\nu-1}$$

$$= x^\nu J_{\nu-1}(x).$$

$$\therefore \boxed{\frac{d}{dx}\left(x^\nu J_\nu(x)\right) = x^\nu J_{\nu-1}(x)}$$

Similarly we can show, $\boxed{\frac{d}{dx}\left(x^{-\nu} J_\nu(x)\right) = -x^{-\nu} J_{\nu+1}(x)}$

$$\Rightarrow \quad J_{\nu+1}(x) + J_{\nu-1}(x) = x^{-\nu}\frac{d}{dx}\left(x^\nu J_\nu(x)\right) - x^\nu \frac{d}{dx}\left(x^{-\nu} J_\nu(x)\right).$$

$$= \nu . x^{-1} J_\nu(x) + \frac{dJ_\nu}{dx} + \nu x^{-1} J_\nu(x) - \frac{dJ_\nu}{dx}.$$

$$= \frac{2\nu}{x} J_\nu(x).$$

$$\Rightarrow \boxed{J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x)}.$$

Similarly, we see, $\boxed{J_{\nu+1}(x) - J_{\nu-1}(x) = 2\frac{dJ_\nu(x)}{dx}}$

The above two equations imply,

$$\boxed{\frac{dJ_\nu(x)}{dx} = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x)}$$

Let $x_{\nu n}$ be a zero of $J_\nu(x)$.

i.e. $J_\nu(x_{\nu n}) = 0$ (for any ν, there are infinite number of roots).

Consider $J_\nu(x_{\nu n}\rho/a)$ as a function of ρ. (It has zeros for $\rho=0,a$).

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(x_{\nu n}\rho/a)}{d\rho}\right) + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}\right)J_\nu(x_{\nu n}\rho/a) = 0 .$$

Multiply by $\rho J_\nu(x_{\nu m}\rho/a)$ and integrate from 0 to a:

$$\int_0^a J_\nu(x_{\nu m}\rho/a)\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(x_{\nu n}\rho/a)}{d\rho}\right)d\rho$$

$$+\int_0^a\left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}\right)\rho J_\nu(x_{\nu m}\rho/a)J_\nu(x_{\nu n}\rho/a)\,d\rho = 0 .$$

$$\Rightarrow -\int_0^a \rho\frac{dJ_\nu(x_{\nu n}\rho/a)}{d\rho}\frac{dJ_\nu(x_{\nu m}\rho/a)}{d\rho} + \int_0^a\left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}\right)\rho J_\nu(x_{\nu m}\rho/a)J_\nu(x_{\nu n}\rho/a)\,d\rho=0$$

interchange $n \leftrightarrow m$ & subtract one from the other:

$$(x_n^2 - x_m^2)\int_0^a \rho J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu m}\rho/a)\,d\rho=0 .$$

$$\Rightarrow \int_0^a \rho J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu m}\rho/a)\,d\rho=0 \quad \text{if } n\neq m.$$

For $n=m$,

$$\int_0^a \rho\left(J_\nu(x_{\nu n}\rho/a)\right)^2 d\rho \neq 0 .$$

To find the normalization, consider

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dJ_\nu}{dx}\right) + \left(1-\frac{\nu^2}{x^2}\right)J_\nu = 0$$

Let $x = \rho\alpha$

$$\Rightarrow \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(\rho\alpha)}{d\rho}\right) + \left(\alpha^2 - \frac{\nu^2}{\rho^2}\right)J_\nu(\rho\alpha) = 0 . \quad]\times J_\nu(\rho\beta)$$

Similarly, if $x=\rho\beta$, we get

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(\rho\beta)}{d\rho}\right) + \left(\beta^2 - \frac{\nu^2}{\rho^2}\right)J_\nu(\rho\beta) = 0 \quad]\times J_\nu(\rho\alpha)$$

$$\Rightarrow \frac{1}{\rho} J_\nu(\rho\beta) \frac{d}{d\rho}\left(\rho \frac{dJ_\nu(\rho\alpha)}{d\rho}\right) - \frac{1}{\rho} J_\nu(\rho\alpha) \frac{d}{d\rho}\left(\rho \frac{dJ_\nu(\rho\beta)}{d\rho}\right) + (\alpha^2-\beta^2) J_\nu(\rho\alpha) J_\nu(\rho\beta) = 0$$

or $$J_\nu(\rho\beta) \frac{d}{d\rho}\left(\rho \frac{dJ_\nu(\rho\alpha)}{d\rho}\right) - J_\nu(\rho\alpha) \frac{d}{d\rho}\left(\rho \frac{dJ_\nu(\rho\beta)}{d\rho}\right) + (\alpha^2-\beta^2) \rho J_\nu(\rho\alpha) J_\nu(\rho\beta) = 0$$

$$\Rightarrow \frac{d}{d\rho}\left(J_\nu(\rho\beta)\rho \frac{dJ_\nu(\rho\alpha)}{d\rho} - J_\nu(\rho\alpha)\rho \frac{dJ_\nu(\rho\beta)}{d\rho}\right) + (\alpha^2-\beta^2)\rho J_\nu(\rho\alpha) J_\nu(\rho\beta) = 0.$$

~~Now, if α=~~

$$\Rightarrow \left(J_\nu(\rho\beta)\rho \frac{dJ_\nu(\rho\alpha)}{d\rho} - J_\nu(\rho\alpha)\rho \frac{dJ_\nu(\rho\beta)}{d\rho}\right) + (\alpha^2-\beta^2)\int d\rho\, \rho J_\nu(\rho\alpha) J_\nu(\rho\beta) = 0$$

set $\beta = \frac{x_{\nu n}}{a}$. $\Rightarrow (J_\nu(a\beta) = 0\ .)$

$$\left(J_\nu(\rho\beta)\rho \frac{dJ_\nu(\rho\alpha)}{d\rho} - J_\nu(\rho\alpha)\rho \frac{dJ_\nu(\rho\beta)}{d\rho}\right)\Bigg|_0^a + \left(\alpha^2 - \frac{x_{\nu n}^2}{a^2}\right)\int_0^a d\rho\, \rho J_\nu(\rho\alpha) J_\nu(\rho x_{\nu n}/a) = 0$$

$$\Rightarrow -J_\nu(a\alpha)\, a \left(\frac{dJ_\nu(\rho\beta)}{d\rho}\right)_{\rho=a,\ \beta=\frac{x_{\nu n}}{a}} + \left(\alpha^2 - \frac{x_{\nu n}^2}{a^2}\right)\int_0^a d\rho\, \rho J_\nu(\rho\alpha) J_\nu\left(\rho \frac{x_{\nu n}}{a}\right) = 0$$

$$\Rightarrow \int_0^a d\rho\, \rho\, J_\nu(\rho\alpha) J_\nu\left(\rho \frac{x_{\nu n}}{a}\right) = \frac{J_\nu(a\alpha)\, a \left(\frac{dJ_\nu(\rho\beta)}{d\rho}\right)_{\rho=a,\ \beta=\frac{x_{\nu n}}{a}}}{\left(\alpha^2 - \frac{x_{\nu n}^2}{a^2}\right)}.$$

Take the limit $\alpha \to \frac{x_{\nu n}}{a}$.

$$\text{R.H.S.} = \frac{\left(\frac{dJ_\nu(a\alpha)}{d\alpha}\right)_{\alpha=\frac{x_{\nu n}}{a}}\, a \left(\frac{dJ_\nu(\rho\beta)}{d\rho}\right)_{\rho=a,\ \beta=\frac{x_{\nu n}}{a}}}{2\cdot \frac{x_{\nu n}}{a}}$$

Use the recurrence relation

$$J_{\nu+1}(x) = \frac{\nu}{x} J_\nu(x) - \frac{dJ_\nu(x)}{dx}.$$

$$\Rightarrow \int_0^a d\rho\, \rho \left(J_\nu\left(\rho\frac{x_{\nu n}}{a}\right)\right)^2 = \frac{a^2}{2}\left(J_{\nu+1}(x_{\nu n})\right)^2.$$

Modified Bessel Function:

$$\frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right) f = 0 .$$

$k \to ik$ in Bessel equation gives this one.

solutions to this equation are

$$I_\nu(x) = e^{-\nu\pi i/2} J_\nu\left(x e^{i\pi/2}\right) .$$

↳ convension.

In terms of the series,

$$I_\nu(x) = \sum_{s=0}^{\infty} \frac{1}{s!\,(s+\nu)!}\left(\frac{x}{2}\right)^{2s+\nu}$$

$$I_{-\nu}(x) = \sum_{s=0}^{\infty} \frac{1}{s!\,(s-\nu)!}\left(\frac{x}{2}\right)^{2s-\nu}$$

For integer ν, $\quad I_\nu(x) = I_{-\nu}(x)$.

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) .$$

Generating function:

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n .$$

$$\text{L.H.S.} = \sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty}(-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}$$

→ substituting $r = n+s$.

$$= \sum_{r,s}\left(\frac{x}{2}\right)^{r+s} \frac{t^{r-s}}{r!\,s!}(-1)^s = \sum_{n,s}\left(\frac{x}{2}\right)^{n+2s} \frac{t^n}{(n+s)!\,s!}(-1)^s$$

$$= \sum_n t^n \left(\sum_{s=0}^{\infty}(-1)^s \frac{1}{s!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s}\right)$$

$$= \sum_n t^n J_n(x) .$$

$$\therefore \quad \boxed{e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n}$$

Integral representation

consider $\quad e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n .$

set $\quad t = e^{i\theta}$.

$\Rightarrow \quad e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta} .$

$$\Rightarrow \quad \int_0^{2\pi} d\theta \; e^{i(x\sin\theta - m\theta)} = \int_0^{2\pi} d\theta \sum_{n=-\infty}^{\infty} J_n(x)\, e^{i(n-m)\theta} .$$

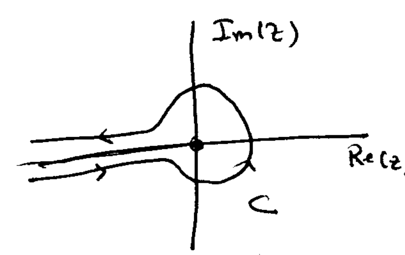
$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \; e^{i(n-m)\theta} = \cancel{2\pi}\,\delta_{nm} .$$

$$\Rightarrow \quad \boxed{J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; e^{i(x\sin\theta - n\theta)}}$$

set $\quad z = e^{i\theta}, \qquad d\theta = \frac{e^{-i\theta}}{i}\,dz = \frac{1}{iz}\,dz .$

$$\therefore \quad e^{i(x\sin\theta - n\theta)}\, d\theta = dz\; e^{x(z-z^{-1})/2}\, z^{n-1}\, \frac{1}{i}$$

$$J_n(x) = \frac{1}{2\pi i}\int_C e^{(x/2)(z-1/z)}\, z^{n-1}\, dz .$$



Rescaling, $\quad z \to z/(x/2)$.

$$J_n(x) = \frac{1}{2\pi i}\int_C e^{(z - x^2/4z)} \cdot \left(\frac{x}{2}\right)^{\nu} z^{n-1}\, dz .$$

$$\Rightarrow \quad \boxed{J_n(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^{\nu}\int_C e^{z - \frac{x^2}{4z}}\, z^{n-1}\, dz .}$$