Complex functions:

→ Every function is defined on an open set.

Limits & continuity:

Def: The function $f(x)$ is said to have the limit 'A' as x tends to a,

$$\lim_{x \to a} f(x) = A,$$

if and only if the following is true:

For every $\epsilon > 0$, there exists a number $\delta > 0$ with the property that $|f(x) - A| < \epsilon \ \forall x$ s.t. $|x-a| < \delta, x \neq a$.

Def: The function $f(x)$ is said to be 'continuous at a' if and only if $\lim_{x \to a} f(x) = f(a)$. A continuous function is one which is continuous at all points where it is defined.

Derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

→ Let $f(z)$ be a real function of a complex variable, whose derivative exists at $z = a$.

The limit of the different quotient must be the same regardless of the way in which h approaches zero

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \rightarrow \text{Real as } h \text{ tends to zero through real values.}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+ih) - f(a)}{ih} \rightarrow \text{Purely imaginary}$$

$\Rightarrow$ $f'(a)$ must be zero.

→ Real function of a complex variable ~~is~~ has the derivative zero, or else the derivative does not exist.

→ Complex function of a real variable.

$$z(t) = x(t) + i\,y(t)$$

$$z'(t) = x'(t) + i\,y'(t) \Rightarrow \text{derivative exists.}$$

Derivative of a complex function of a complex variable:

Analytic function / holomorphic function } complex function of a complex variable which possess a derivative wherever the function is defined.

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

~~If which~~ If we choose $h =$ real, imaginary part y remains constant.

→ Partial derivative w.r.t. x.

$f(z) = u(x,y) + i\,v(x,y)$.

$$\therefore \quad f'(z) = \frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$h =$ pure imaginary: $h = ik$

$$f'(z) = \lim_{k\to 0} \frac{f(z+ik) - f(z)}{ik}$$

$$= -i \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k} = -i\frac{\partial f}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

$\therefore$ We must have,

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$ → Cauchy-Riemann conditions.

→ The derivative of an analytic function is itself analytic.

⇒ $\nabla^2 u = 0 = \nabla^2 v$. → Laplace's equation.

u, v → Harmonic.

The real and imaginary part of an analytic function ... harmonic.

→ If $u(x,y)$ and $v(x,y)$ have continuous first order partial derivatives which satisfy the Cauchy-Riemann differential equations, then $f(z) = u(z) + i v(z)$ is analytic with continuous derivative $f'(z)$, and conversely.

$$\left.\begin{aligned} z &= x+iy, \\ \bar{z} &= x-iy \end{aligned}\right\} \rightarrow \quad x = \frac{1}{2}(z+\bar{z}), \quad y = \frac{1}{2i}(z-\bar{z}).$$

Treat $z, \bar{z}$ as independent variable.

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}(u+iv)\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ &= 0. \quad \text{if } f \text{ is analytic.} \end{aligned}$$

→ Every polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is an analytic function.

$P(z)$: Polynomial of degree 'n'.

If $P(\alpha_1) = 0$, then $P(z) = (z-\alpha_1)\, P_1(z)$. $\quad P_1(z)$: Polynomial of degree $(n-1)$.

Similarly, $P(z) = a_n (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$.

$\alpha_1, \ldots \alpha_n$ are not necessarily distinct.

if h of the α_i's coincide

$\longrightarrow \alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_h} = \alpha^*$.

'α^*' is a zero of ~~$p(z)$~~ order h of $p(z)$.

Suppose α is a zero of order h.

Then $p(z) = (z-\alpha)^h p_h(z)$, s.t. $p_h(\alpha) \neq 0$.

$\Rightarrow p(\alpha) = p'(\alpha) = \cdots p^{(h-1)}(\alpha) = 0, \quad p^{h}(\alpha) \neq 0$.

$\rightarrow$ The order of a zero equals ~~to~~ the order of first nonvanishing derivative.

$\rightarrow$ A zero of order 1 = simple zero.