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Cauchy's Theorem:

If $f(z)$ is analytic within and on a closed contour C, then

$$\oint_C f(z)\, dz = 0 .$$

Cauchy Integral formula:

If $f(z)$ is analytic within and on a closed contour C, then

$$\oint_C \frac{f(z)}{z-a}\, dz = \begin{cases} 2\pi i\, f(a) & \text{if } a \text{ is interior to } C \\ 0 & \text{if } a \text{ is exterior to } C . \end{cases}$$

Goursat formula:

If $f(z)$ is analytic within and on a closed contour C, (and if 'a' is interior to C), then

$$\left(\frac{d^n f}{dz^n}\right)_{z=a} = \frac{n!}{(2\pi i)} \oint \frac{f(z)}{(z-a)^{n+1}}\, dz \quad , \quad n=0,1,\cdots$$

Taylor series expansion:

If $f(z)$ is analytic within and on the circle C of radius r around $z=a$, then there exists a unique and uniformly convergent series in powers of $(z-a)$:

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k \quad , \quad \left(|z-a| \le r \right),$$

with

$$c_k = \frac{1}{k!}\left(\frac{d^k f}{dz^k}\right)_{z=a} = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-a)^{k+1}}\, dw .$$

Laurent series expansions:

Let $f(z)$ be analytic within and on a closed contour C except at a point $z=a$ enclosed by C. Then, $f(z)$ can be expanded around $z=a$ as

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-a)^n ,$$

with the definition

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-a)^{n+1}}\, dw .$$

Singularity: A singularity of a complex function $f(z)$ is any point where it is not analytic. In particular, the point $z=a$ is called an isolated singularity if and only if $f(z)$ is analytic in some neighbourhood but not at $z=a$.

An isolated singularity is

(i) a removable singularity if and only if $f(z)$ is finite throughout a neighborhood of $z=a$, except possibly at $z=a$ itself.

(ii) a pole of order m $(m=1,2,\cdots)$ if and only if $(z-a)^m f(z)$ is analytic at $z=a$ but not $(z-a)^{m-1} f(z)$

(iii) an essential singularity if and only if the Laurent series of $f(z)$ has infinite number of terms involving negative powers of z.

Examples:

$\frac{\sin z}{z}$ has a removable singularity at $z=0$.

$\frac{1}{z}$ has a pole of order 1 (simple pole) at $z=0$

$e^{1/z}$ has an essential singularity at $z=0$.

Rational Functions:

$$R(z) = \frac{P(z)}{Q(z)} \quad ; \quad \text{quotient of two polynomials.}$$

$\rightarrow$ $P(z), Q(z)$ has no common factors.

$\rightarrow$ $R(z)$ is infinite at zeros of $Q(z)$.

$\rightarrow$ Rational functions are functions valued ~~at~~ in the extended complex plane.

$\rightarrow$ zeros of $Q(z)$: poles of $R(z)$.

order of a pole of $R(z)$ is defined to be the order of the ~~zero~~ corresponding zero of $Q(z)$.

The derivative $$R'(z) = \frac{P'(z)\,Q(z) - Q'(z)\,P(z)}{Q^2(z)}$$

exists only when $Q(z) \neq 0$.

$\rightarrow$ $R'(z)$ has the same poles as $R(z)$

$\rightarrow$ order of each pole is increased by one.

* Let z & $R(z)$ are valued in the extended complex plane

Define $R_1(z) = R(1/z)$.

$\rightarrow$ order of zero or pole of $R(z)$ at $z=\infty$ is defined as the order of zero or pole of $R_1(z)$ at $z=0$.

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

$$\Rightarrow \quad R_1(z) = z^{m-n}\left(\frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}\right)$$

If $m > n$, $R(z)$ has a zero of order $(m-n)$ at ∞.

If $m < n$, $R(z)$ has a pole of order $(m-n)$ at ∞.

Residue Theorem:

If $f(z)$ is analytic everywhere within a closed contour C except at a finite number of poles, its contour integral along C yields

$$\oint_C f(z)\, dz = 2\pi i \sum_j \text{Res}(f, a_j).$$

$\text{Res}(f, a_j)$ is the residue of $f(z)$ at the pole $z = a_j$.

When the pole $z = a_j$ is mth order,

$$\text{Res}(f, a_j) = \frac{1}{(m-1)!} \lim_{z \to a_j} \left(\frac{d}{dz}\right)^{m-1} \left((z - a_j)^{m-1} f(z) \right).$$

Principal value Integral:

The notation

$$P \int_{-R}^{R} \frac{f(x)}{x - \alpha}\, dx \equiv \lim_{r \to 0} \left[\int_{-R}^{\alpha - r} \frac{f(x)}{x - \alpha}\, dx + \int_{\alpha + r}^{R} \frac{f(x)}{x - \alpha}\, dx \right]$$

provides the principal value integral of $\frac{f(z)}{z - \alpha}$ for real α.

If $f(z)$ is analytic in the upper half plane, and vanishes sufficiently rapidly as $|z| \to \infty$, then

$$\lim_{R \to \infty} P \int_{-R}^{R} \frac{f(x)}{x - \alpha}\, dx = i\pi f(\alpha).$$