Note on Fourier Series: (Shima & Nakayama)

A "trigonometric series"

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

is called a "Fourier series" of a function $f(x)$ iff

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx .$$

If the Fourier series converges to $f(x)$ at all points $x \in [0, 2\pi]$ you can write the equality $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. ≡ Uniform convergence

Q. When the series converges? (sufficient condition).

(A) If 1. $f(x)$ is continuous in $[0, 2\pi]$
2. $f(x)$ is periodic with period 2π
3. $\frac{df}{dx}$ is continuous or at most has finite number of discontinuity in $[0, 2\pi]$.

then the Fourier series of $f(x)$ converges uniformly to $f(x)$.

(B) If 1. $f(x)$ is periodic in $[0, 2\pi]$ (with period 2π), but not continuous.
2. df/dx has finite number of discontinuity in $[0, 2\pi]$

then, the Fourier series of $f(x)$
a) converges to $f(x)$ if x is a point of continuity.
b) converges to $\frac{1}{2} \left(\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \right)$ where x_0 is a point of discontinuity.

[Ps: These are sufficient conditions, not necessary conditions].

Setting, $c_n = \frac{1}{2}(a_n - i b_n)$, $c_{-n} = c_n^*$, we find

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad ; \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx .$$

Suppose $f(x)$ is periodic with period λ (and satisfies the other sufficient conditions mentioned earlier).

Then $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{\lambda}\right) + b_n \sin\left(\frac{2\pi n x}{\lambda}\right) \right].$$

$$a_n = \frac{2}{\lambda} \int_0^{\lambda} f(x) \cos\left(\frac{2\pi n x}{\lambda}\right) dx$$

$$b_n = \frac{2}{\lambda} \int_0^{\lambda} f(x) \sin\left(\frac{2\pi n x}{\lambda}\right) dx .$$

setting $c_n = \frac{1}{2}(a_n - i b_n)$, $n > 0$ & $c_{-n} = c_n^*$ we find

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/\lambda}$$

where $$c_n = \frac{1}{\lambda} \int_0^{\lambda} f(x) e^{-2\pi i n x/\lambda} dx .$$

- If $f(x)$ is periodic in $[-\lambda/2, \lambda/2]$ and is even $(f(x) = f(-x))$

then, $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{\lambda}\right)$$

with $$a_n = \frac{4}{\lambda} \int_0^{\lambda/2} f(x) \cos\left(\frac{2\pi n x}{\lambda}\right) dx .$$

- If $f(x)$ is periodic in $[-\lambda/2, \lambda/2]$ and is odd, $(f(x) = -f(-x))$

then $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{\lambda}\right)$$

where $$b_n = \frac{4}{\lambda} \int_0^{\lambda/2} f(x) \sin\left(\frac{2\pi n x}{\lambda}\right) dx .$$

Fourier series of non-periodic functions:

Let $f(x)$ is a function in $[0, L]$ ($f(x)$ is not periodic).

Define $f_e(x)$ in $[-L, L]$,

such that
$$f_e(x) = \begin{cases} f(x) & \text{in } x \in [0, L] \\ f(-x) & \text{in } x \in [-L, 0] . \end{cases}$$

$f_e(x)$ is periodic in $[-L, L]$ ∴ can use the standard Fourier series to express $f_e(x)$ in $[-L, L]$. Same series can be used for $f(x)$ in $[0, L]$.

You can also use
$$f_o(x) = \begin{cases} f(x) & \text{in } x \in [0, L] \\ -f(-x) & \text{in } x \in [-L, 0] . \end{cases}$$

Fourier series in 3D:

$$f(\vec{r}) = \sum_{\vec{k}} C_{\vec{k}} \, e^{i \vec{k} \cdot \vec{r}} , \qquad \vec{k} = 2\pi \left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3} \right).$$

where
$$C_{\vec{k}} = \frac{1}{V} \int_V f(\vec{r}) \, e^{-i \vec{k} \cdot \vec{r}} \, d^3 r .$$

$$V = L_1 L_2 L_3 .$$

Mean convergenie:

$$\lim_{N \to \infty} \int_0^{\lambda} \left| f(x) - \sum_{n=-N}^{N} c_n e^{i 2\pi n x / \lambda} \right|^2 dx = 0 .$$

. A necessary and sufficient condition for mean convergenie is
$$\frac{1}{\lambda} \int |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 .$$
(Parseval Identity).